THE PROOF SOCIETY

AUTUMN SCHOOL AND WORKSHOP (TPS2022)

7-12 November 2022, Utrecht, The Netherlands

Evert Willem Beth Foundation

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ASL
Association for Symbolic Logic

Utrecht University

University of Amsterdam

Programme

4th International Autumn School for Proof Theory

	9:30	9:50	10:00	11:00	11:30	12:30	14:30	15:30	16:00	17:30
Monday	Registration	Opening	1A	coffee	2A	lunch	3A	coffee	1B	
Tuesday			2B	coffee	3B	lunch	1C	coffee	2C	Reception
Wednesday			4A	coffee	3C	lunch	5A	coffee	4B	
Thursday			5B	coffee	4C	lunch	5C	coffee	TPS AGM	

1: Tutorial on proof-theoretic semantics — Bogdan Dicher

2: Labelled sequent calculi — Marianna Girlando

3: Proof Complexity — Raheleh Jalali

4: Computational Content of Proofs — Ulrich Berger

5: Proof Theory of Set Theory — Michael Rathjen

4th Workshop on Proof Theory and Its Applications

	8:30	9:15	10:00	10:30	11:15	12:00	13:30	14:15	15:30	16:00	17:15	19:30
Friday	I	C1	coffee	II	C2	lunch	III	C3	coffee	C4		dinner
Saturday	IV	C5	coffee	V	C6	lunch	VI	C7	coffee	C8	award ceremony and closing	

I: Proof-theoretic analysis of automated inductive theorem proving
— Stefan Hetzl

II: Explicit Mathematics Reloaded
— Gerhard Jäger

III: Herbrand meets cyclic proofs
— Sebastian Enqvist

IV: Reductio ad absurdum
— Hajime Ishihara

V: Modal logic and the polynomial hierarchy
— Sonia Marin

VI: Converse extensionality and apartness
— Benno van den Berg

C1
Sofía Santiago-Fernández
Amirhossein Akbar Tabatabai
Fabio De Martin Polo

C2
Alexandra Pavlova
Matteo Acclavio
Marianela Morales

C3
Lukas Melgaard
Jan Rooduijn
Johannes Kloibhofer
Gianluca Curzi
Marianna Girlando

C4
Joost J. Joosten
Juan Miguel Lopez Munive
Antonio Piccolomini d'Aragona
Masanobu Toyooka
Takahiro Yamada

C5
Will Stafford
Philipp Provenzano
Giovanni Soldà

C6
Avgerinos Delkos
Paulo Guilherme Santos
Anupam Das

C7
Masaya Taniguchi
Morenikeji Neri
Cosimo Perini Brogi
Agata Tomczyk
Iris van der Giessen

C8
David M. Cerna
Matthias Baaz
Juan Aguilera
Fedor Pakhomov

Programme Committee: Bahareh Afshari, Matthias Baaz, Arnold Beckmann, Lev Beklemishev, Balthasar Grabmayr, Rosalie Iemhoff, Joost Joosten, Antonina Kolokolova, Norbert Preining, and Andreas Weiermann.

Local Organising Committee: Bahareh Afshari, Saul Gebhardt, Sophie Glade, Menno Hellinga, Rosalie Iemhoff, Raheleh Jalali, Robin Martinot, Thierry Orth, and Robert Passmann.

Contents

Invited speakers

Contributed talks (listed by speaker)

Invited Speakers

Converse extensionality and apartness

Benno van den Berg[1,*], **Robert Passmann**[2]
[1]*Institute for Logic, Language and Computation, University of Amsterdam, The Netherlands*
[2]*Institute for Logic, Language and Computation, University of Amsterdam, The Netherlands*
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Following Kreisel one of the main concerns of proof theory has become the extraction of hidden computational information from proofs. For this purpose Gödel's Dialectica interpretation (combined with negative translation, if necessary) has proven itself to be indispensable (see [2]).

One of the hardest principles to interpret using a functional interpretation is the principle of function extensionality. The reason for this is that the Dialectica interpretation requires one to interpret a stronger form of extensionality, which we have dubbed *converse extensionality*.

In this talk I explain how one can give a computational interpretation of this principle using Brouwer's notion of *apartness*. Brouwer's idea was that equality might not be primitive concept and could be defined as the negation of a strong notion of inequality called apartness. The paradigmatic example is the real numbers, where two reals r and s are apart when there are disjoint intervals with rational endpoints I_1 and I_2 such that $r \in I_1$ and $s \in I_2$. Equality of real numbers can then be defined as not being apart.

This talk is based on the preprint [1] written together with Robert Passmann. In that paper we heavily use the language of category theory. However, in this talk I will explain our results in proof-theoretic terms – no knowledge of category theory will be required.

References

[1] B. van den Berg and R. Passmann. Converse extensionality and apartness. arXiv:2103.14482, 2021.

[2] U. Kohlenbach. *Applied proof theory: proof interpretations and their use in mathematics*, Springer-Verlag, Berlin, 2008.

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Computational Content of Proofs

Ulrich Berger[1,*]
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One often says that from a constructive mathematical proof of a theorem one can read-off an algorithm provided one has a representation of the mathematical objects involved. For example, the Intermediate Value Theorem states that every continuous function on the reals that is negative at 0 and positive at 1 has a zero between 0 and 1. With some extra conditions on the function the proof of the IVT can be done constructively and an algorithm for finding the zero can be read-off. This presupposes a representation of (exact) real numbers and a representation of continuous functions (with extra conditions). The example suggests that this 'reading-off' process is rather ad hoc and heavily dependent on the particular problem area the theorem is about. In fact this is not so. I will present a uniform theory, based on historic work by Kleene and Kreisel, for reading-off (or 'extracting') the computational content of proofs that applies to a wide range of abstract axiomatic mathematics. The theory automatically provides representations of mathematical objects and formal proofs of the correctness of the extracted algorithms.

The tutorial will cover: Historical origins, formal definitions of various extraction methods, discussion of current proof assistants supporting the extraction of the computational content of proofs, main application areas, and concrete examples and practical program extraction exercises.

Tutorial on Proof-theoretic Semantics

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Proof-theoretic semantics (PTS) is an inferentialist theory of meaning which originates in the work of Gentzen in the 1930s and was subsequently developed by Prawitz, Martin-Löf, Dummett, and more recently by Schroeder-Heister, who also baptised the theory, and many others. It is an alternative to Tarskian model-theoretic semantics, aiming to explain the meaning of the logical constants in terms of the rules of inference that govern their behaviour in proofs.

The *orthodox* version of PTS, developed against the background of natural deduction, can be described as an extended attempt to develop Gentzen's suggestion that 'the introduction [rules] represent, as it were, the 'definitions' of the [logical constants], and the eliminations are no more [...] than the consequences of these definitions'. At its core lies the notion of *harmony*: a kind of balance between the relative strength of the introductions and eliminations of a logical constant that testifies to their successfully defining a logical constant. The quest for a formal property that accurately captures the intuitive notion of harmony has dominated much of PTS. Said quest is the source of *less orthodox* versions of PTS. By and large, these retain the focus on harmony, while taking revisionary stances with respect to other aspects of orthodox PTS, such as the priority of the standard assertionist setting of PTS, or of natural deduction.

The first part of the tutorial will critically discuss orthodox PTS, focusing on the development of different conceptions of harmony and their connection with formal properties of Gentzen-style calculi, such as reducibility, invertibility and normalizability or cut-elimination.

The second part of the tutorial is devoted to less orthodox stances in PTS. We will first look at bilateralist versions of the programme, which put denial on a par with assertion and thus introduce a new dimension of harmony, between the conditions for asserting and, respectively, denying a sentence. Finally, we will look at versions of PTS that take the sequent calculus as the framework of choice for specifying definitional rules for the logical connectives. We will explore the motivation(s) for going down this path and some results obtained within this framework.

A detailed list of the topics of the tutorial, including recommended readings, is available at `bdicher.me` *in the section* PSAST*.*

Herbrand meets cyclic proofs

Sebastian Enqvist[1,*], **Bahareh Afshari, Graham E. Leigh**
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Classical logic, in contrast with intuitionistic logic, does not have the *existence property*. One may classically prove an existentially quantified statement without necessarily providing an explicit witness. However, *Herbrand's theorem* states a weaker form of the existence property for classical logic: if a prenex Σ_1-formula $\exists\vec{x}\varphi(\vec{x})$ is valid then there is a finite set of tuples of terms $\{\vec{t_1}, \ldots, \vec{t_n}\}$ (a *Herbrand set*) such that the disjunction $\varphi(\vec{t_1}) \vee \cdots \vee \varphi(\vec{t_n})$ is valid. Herbrand's theorem can be proved via cut elimination, and supplies a form of "computational content" to classical proofs. A more direct representation of this computational content was provided in recent work [1], where each proof is associated with a non-deterministic *higher-order recursion scheme*. These so-called *Herbrand schemes* can be seen as abstract representations of non-deterministic programs that extract witnesses for existentially quantified formulas. In particular, when the end sequent is prenex Σ_1, the scheme associated with a proof provides a Herbrand set.

Since first-order predicate logic does not contain any explicit mechanism for inductive reasoning, the recursion scheme associated with a proof will be *acyclic*, meaning that the programs that we can represent in this manner will be quite limited as they contain no recursive function calls. An approach to overcome the limitation, akin to adding induction, is to employ the system of *cyclic proofs* for first-order logic extended with Martin-Löf style inductive definitions, introduced by Brotherston and Simpson [2]. In this talk we show how Herbrand schemes can be defined for cyclic proofs, and give examples demonstrating how such schemes can be used to provide computational content of inductive proofs in a classical setting.

References

[1] B. Afshari, S. Hetzl and G. Leigh, Herbrand's theorem as higher-order recursion, *Annals of Pure and Applied Logic* **171** (2020).

[2] J. Brotherston and A. Simpson, Cyclic proof for first-order logic with inductive definitions, in *Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX 2005)* pp. 78–92.

Labelled Sequent Calculi

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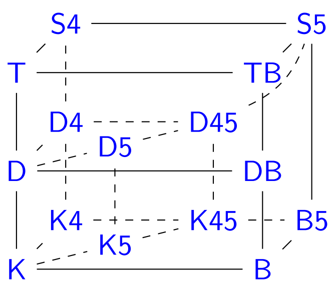
This tutorial aims at presenting a general recipe for defining labelled sequent calculi on the basis of classes of possible-worlds models. Moreover, we will introduce the general strategies needed to prove soundness, completeness, cut-admissibility and other structural properties of the calculi. Modal logics will be our main case study.

Several researches have studied labelled proof systems, including Fitting [1], Gabbay [2], Viganò [3] and Negri [4,5]. Labelled sequent calculi enrich the language of standard Gentzen-style calculi by syntactic elements, the 'labels', which represent pieces of information from the semantics of the logic under scope. Thus, labelled calculi represent an expressive and versatile formalism, allowing to define analytic proof systems for a wide variety of logics, including modal logics [4] and intermediate logics [5,6,7].



k	$\Box(A \to B) \to (\Box A \to \Box B)$
nec	*if* $\vdash A$ *then* $\vdash \Box A$
d	$\Box A \to \Diamond A$
t	$A \to \Diamond A$
b	$A \to \Box\Diamond A$
4	$\Diamond\Diamond A \to \Diamond A$
5	$\Diamond A \to \Box\Diamond A$

Figure 1: The S5 cube of modal logics and their axiom systems

The tutorial is structured as follows. After introducing Gentzen style sequent calculi and possible-worlds semantics, we will present modular labelled calculi for logics in the S5 cube, displayed in Figure 1, and briefly discuss how they relate to other kinds of calculi for modal logics, e.g., hypersequents [8] and nested sequents [9]. We will then show how to prove soundness and cut-admissibility for the labelled calculi introduced, and how to prove completeness by extracting a countermodel from a failed proof search tree, following the method from [10]. Finally, depending on time, we will introduce labelled calculi for conditional logics, which are modal logics having a simple topological semantics [11], and for intuitionistic (modal) logics, whose semantics is defined in terms of (bi)relational possible-worlds models [5,12,13].

References

[1] M. Fitting, *Proof methods for modal and intuitionistic logics*, Vol. 169, Springer Science & Business Media, 2013.

[2] D. M. Gabbay, *Labelled Deductive Systems*, Oxford Logic Guides, Clarendon Press, Oxford, 1996.

[3] L. Viganò, *Labelled Non-Classical Logics*, Springer Science & Business Media, Switzerland, 2000.

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[7] A. Ciabattoni, P. Maffezioli, and L. Spendier. Hypersequent and labelled calculi for intermediate logics, in: *International Conference on Automated Reasoning with Analytic Tableaux and Related Methods*, Springer, Berlin, Heidelberg, 2013.

[8] A. Avron, The method of hypersequents in the proof theory of propositional non-classical logics, in: *Logic: from foundations to applications*, Clarendon, 1996.

[9] K. Brünnler, Deep sequent systems for modal logic, in: *Archive for Mathematical Logic*, 48(6), pp.551–577, 2009.

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[11] M. Girlando, S. Negri, and N. Olivetti, Uniform labelled calculi for preferential conditional logics based on neighbourhood semantics, in: *Journal of Logic and Computation*, 31.3, pp. 947-997, 2021.

[12] P. Maffezioli, A. Naibo, and S. Negri, The Church–Fitch knowability paradox in the light of structural proof theory, in: *Synthese*, 190(14), pp. 2677–2716, 2013.

[13] S. Marin, M. Morales, and L. Straßburger, A fully labelled proof system for intuitionistic modal logics, in: *Journal of Logic and Computation*, Oxford University Press (OUP), 2021.

Proof-theoretic analysis of automated inductive theorem proving

Stefan Hetzl[1,*], **Jannik Vierling**[1]

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Automating the search for proofs by induction is an imporant topic in computer science with a history that stretches back decades. A variety of different approaches and systems has been developed. Typically, these systems have been evaluated empirically and thus very little is known about their theoretical limitations.

In this talk I will present a proof-theoretic approach for understanding the power and limits of methods for automated inductive theorem proving. A central tool are translations of proof systems that are intended for automated proof search into (very) weak arithmetical theories. This allows not only to locate a method in a partial order of theories but also to provide examples for unprovable statements which are of practical interest in computer science.

I will describe concrete such proof-theoretic analyses of two methods of automated inductive theorem proving: adding explicit induction axioms to a saturation theorem prover [1] and clause set cycles [2].

References

[1] Stefan Hetzl and Jannik Vierling. Induction and Skolemization in saturation theorem proving. *Annals of Pure and Applied Logic*, 174(1):103167, 2023.

[2] Stefan Hetzl and Jannik Vierling. Unprovability results for clause set cycles. *Theoretical Computer Science*. to appear.

[3] Jannik Vierling. *The limits of automated inductive theorem provers.* PhD thesis, TU Wien, Austria.

Reductio ad absurdum

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Many classical mathematicians consider constructive mathematics a mathematics without *reductio ad absurdum* (RAA). However, it seems that there is no firm consensus between classical and constructive mathematicians on what is RAA, and this leads classical mathematicians to misunderstand constructive mathematics.

In this talk, we clarify what is RAA from constructive point of view and a typical misunderstanding of classical mathematicians on RAA. We also examine the definition and an example of RAA in a Japanese high school mathematics text book, where RAA is called "背理法 (hairihō)". It appears that the definition is acceptable from constructive point of view, but the example not.

References

[1] T. Ohshima et al., 数学 I (*Mathematics I*), Sūkenshuppan, 2014.

[2] A. S. Troelstra and D. van Dalen, *Constructivism in mathematics. Vol. I. An introduction*, Studies in Logic and the Foundations of Mathematics, 121, North-Holland Publishing Co., Amsterdam, 1988.

[3] D. van Dalen, *Logic and Structure*, 5th ed., Springer, London, 2013.

Introduction to Proof Complexity

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"A student of mine asked me today to give him a reason for a fact which I did not know was a fact— and do not yet. He says that if a figure be anyhow divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured—four colours may be wanted, but not more .... If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did ..." [3]

This is what the famed mathematician De Morgan wrote to his friend, Hamilton, the distinguished mathematician and physicist, in 1852. The content of this letter was the birth of the famous "four color theorem". Over the years, several fallacious proofs were given until finally in 1977, Appel and Haken presented a correct one. The proof, however, required analyzing many (to be precise, 1936) discrete cases. Facing such a tedious case-checking work, the question arises whether there exists a shorter, more brilliant proof. Or, we may more generally wonder:

How hard is it to prove some given theorems? What are their *shortest* proofs? Are there such *hard* theorems that even their shortest proofs go beyond our physical capacities?

Even in the case that we consider the propositional level, these problems are meaningful: Let φ be a classical propositional tautology. By the so-called *brute-force* method, we know that φ has a proof roughly of the size 2^n, where n is the number of the atomic variables in φ. The question is whether there exists a smarter strategy to verify the validity of φ, which does not include checking all the possible valuations.

The problems we mentioned so far focus on theorems rather than the theories in which they are proved. Looking in this direction, one can ask whether there is a theory so strong that no hard theorems exist in it. Otherwise, what happens if there exists no such theory? Then, we may wonder if there will be a significant decrease in lengths of proofs when we move to more powerful theories. If so, we can continue to advance towards stronger and stronger theories and ask: Is there a "strongest" theory, in the sense that it provides the *best* proofs?

These are some examples of the problems considered in *proof complexity*, a field whose main aim is investigating the complexity (for instance, length, i.e., number of symbols) of proofs.

In this course, we begin with introducing proof systems such as Frege and extended Frege systems and resolution. We introduce Cook's program and consider open problems and how they are related to the complexity classes P, NP, and coNP, and in general to the field of computational complexity. We will then talk about interpolation, specially feasible interpolation as one of the main methods to prove lower bounds. Finally, we will talk about the complexity of proofs in systems for non-classical logics. For more, see [1,2].

References

[1] J. Krajíček, *Proof complexity*, Vol. 170. Cambridge University Press, 2019.

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[3] R. Wilson, *Four Colors Suffice: How the Map Problem Was Solved* -Revised Color Edition. Vol. 30. Princeton university press, 2013.

Explicit Mathematics Reloaded

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When Feferman introduced Explicit Mathematics in the mid 1970s his principal aim was to set up a natural formal framework for Bishop's constructive mathematics. But soon it became evident that systems of explicit mathematics (now based on intuitionistic or classical logic) play also an important role in many other parts of logic (such as, for example, subsystems of second order arithmetic and set theory, reductive proof theory, generalized/abstract recursion theory).

In recent years, however, explicit mathematics has become somewhat quiet. In this talk I will come back to explicit mathematics. Not to explicit mathematics in its original form, but to some modifications and extensions that have been triggered by recent developments. In particular, I will discuss:

- The unique ontology of such explicit systems.
- Extensions by universes and/or ordinals.
- New aspects of predicativity and metapredicativity.

References

[1] S. Feferman, A language and axioms for explicit mathematics, in *Algebra and Logic*, J. Crossley (ed.), vol. 450 of Lecture Notes in Mathematics, Springer, 1975, pp. 87–139.

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[3] G. Jäger, Identity, equality, and extensionality in explicit mathematics, to appear in *Handbook of Bishop Constructive Mathematics*, D.S Bridges, H. Ishihara, M. Rathjen, H. Schwichtenberg (eds.).

Modal logic and the polynomial hierarchy[1]

Sonia Marin[1,*]
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Ladner's seminal work [1] showed that a large number of modal logics between K and $S4$ are **PSPACE**-complete. Adding further axioms, such as 5, can simplify the underlying complexity of the validity problem, with $S5$ being *co***NP**-complete. Indeed, the 'gap' between *co***NP**-complete and **PSPACE**-complete normal modal logics has formed the subject of several works in recent years. Yet, we are not aware of attempts to characterise modal fragments corresponding to levels of the polynomial hierarchy (**PH**).

PH essentially delineates **PSPACE** according to 'bounded quantifier alternation' by identifying **PSPACE** with the set of true quantified Boolean formulas (QBFs). There are many translations from QBFs to the basic normal modal logic K, used in particular in modal satisfiability solving, but their commonly employed complexity measures do not match up: in modal solving the key measure is that of *modal depth*; for QBFs it is *quantifier complexity*, i.e. the number of alternations of $\exists$ and $\forall$ in a (prenex) QBF. It is well-known that the alternation of quantifiers in QBFs corresponds precisely with the levels of the polynomial hierarchy. On the other hand, Halpern has showed that the validity problem for K for formulas with modal depth bounded by some constant $d \geq 2$ is in fact only *co***NP**-complete [2] .

In this work, we identify a measure on modal formulas that coincides with quantifier complexity for QBFs. It measures the complexity of proof search in a simple sequent calculus for modal logic K in terms of alternation between *nondeterministic* (modal) rules and *co-nondeterministic* ($\wedge$) rules. It can be seen as a way to provide a formal analogue of 'quantifier complexity' in modal logic and allows us to classify formulas of K into fragments complete for each level of **PH**, with respect to validity.

This result builds on recent work achieving similar delineations for multiplicative additive linear/affine logic [3,4] and fragments of intuitionistic logic [5], also well-known **PSPACE**-complete logics. Although for modal logic K, it is enough to study proof search in a standard cut-free sequent calculus rather than a more sophisticated *focussed system*.

[1] This abstract is based on some joint work with Anupam Das, partially published in the proceedings of *AiML 2022* as *Modal logic K and the polynomial hierarchy: from QBFs to K and back again.*

References

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[3] Anupam Das, Focussing, MALL and the Polynomial Hierarchy, in *Proceedings of the 9th International Joint Conference on Automated Reasoning, Oxford, UK, 14-17 July 2018*, pp. 689–705.

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Proof Theory of Set Theory

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Until the 1970s, proof theoretic investigations were mainly concerned with theories of arithmetic, inductive definitions, subsystems of second order arithmetic, and finite type systems. In the late 1970s and early 1980s, the focus switched to set theories, furnishing ordinal-theoretic proof theory with a uniform and elegant framework.[1]

More recently it was shown that these tools can even be adapted to the context of strong axioms such as the powerset axiom, where one does not attain complete cut elimination but can nevertheless extract witnessing information and characterize the strength of the theory in terms of provable heights of the cumulative hierarchy.

These techniques have interesting applications. For instance, it turns out that Power Kripke-Platek set theory plus the global axiom of choice is conservative for $\Sigma^{\mathcal{P}}$-formulae over the theory without global choice; by contrast, if one adds $V = L$ one gets a massively stronger theory.

Other important application concern the existence property for intuitionic set theories, especially Constructive Zermelo-Fraenkel set theory and Intuitionistic Zermelo-Fraenkel set theory with bounded separation.

The tutorial will (gently) introduce infinitary proof calculi for set theories and develop techniques for cut elimination therein, such as the tool of collapsing derivations. Later, the tutorial will turn to the aforementioned applications.

[1]Especially in the pioneering work of Gerhard Jäger.

Contributed Talks

On Proof Equivalence via Combinatorial Proofs

Matteo Acclavio[1], **Lutz Straßburger**[2]
[1] *University Roma Tre, Rome, Italy*
[2] *INRIA-Saclay, Paris, France*

Proof theory is the branch of logic studying proofs as mathematical objects and it plays an important role in many areas of computer science. The proof theory has seen enormous progress during last century, and many different proof systems have been developed to study specific aspects of logics, their proofs and their compositionality.

The existence of many different definitions for the same mathematical objects is not something unusual. However, unlike in other fields, there is not a clear understanding when two objects defined in two different formalisms are the same, or even when two objects expressed in the same formalisms.

In fact, one can say that proof theory, in its current form, is not the theory of proofs but the theory of proof systems and their properties: theorems of proof theory, like soundness, completeness, cut admissibility, proof complexity, or focusing, are not about proofs but about proof systems.

The standard proof theoretical approach to proof equivalence is inspired by the Curry-Howard correspondence where proofs are interpreted as programs and the normalization procedure as program execution: two proofs are considered to be the same if they have the same normal form. Nevertheless, this approach has some clear limits. On the viewpoint of logic programming, where proofs are interpreted as program executions rather than programs to be executed, this approach makes little sense since proofs are already in normal form. Moreover, for classical logic and its extensions, using the Lafont counter-example [6] it is possible to identify all proofs of the same formula.

An alternative approach to proof identity is based on the rule permutations, where two proofs are considered the same if they can be transformed into each other by a series of simple local transformations permuting the order of rules. Of course this approach requires two proofs to be presented in the same proof system or, in case they are expressed in two different formalisms, that the translation from a formalism into another translates objects which are considered to be equivalent into equivalent objects.

One of the novelties of linear logic, was the use of the graphical syntax of proof nets to represent proofs. This syntax allows to represent proofs as graphs where edges carry the information about formulas while nodes their interaction via connectives or axioms. This representation of proofs in the multiplicative linear logic identified modulo independent rules permutations by the same syntactical object.

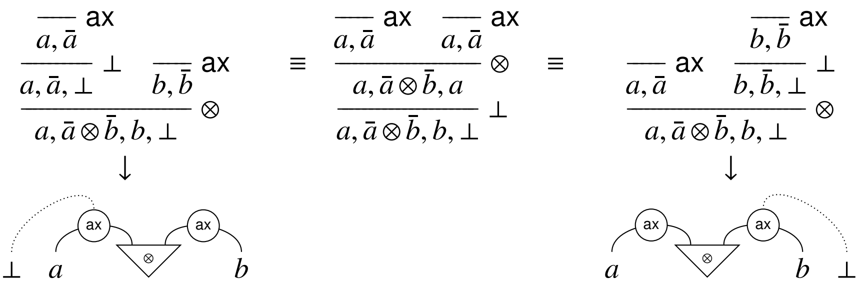

Figure 1: Three derivations which are equivalent modulo rules permutations and two (distinct) proof nets encoding them.

This approach has its limit (as showed in [7, 1]) due to the complexity of the chosen proof equivalence: for some logics it is not possible (unless P-time = P-space) to have a syntax which, at the same time,

1. represents equivalent proofs by the same syntactic object, and
2. is provided with a polynomial-time procedure to check that a syntactic object is the representation of a proof, that is, the syntax is a proof system in the sense of [5].

In this talk we discuss the syntax of *combinatorial proofs* for classical logic [9, 10], a proof system (see 2) capturing a proof equivalence (see 1) for this logic.

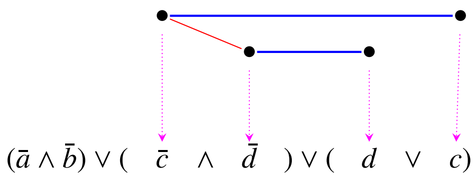

Figure 2: A combinatorial proof of the formula $(\bar{a} \wedge \bar{b}) \vee (\bar{c} \vee \bar{d}) \vee (d \vee c)$

We then analyze the notion of proof equivalence it enforces in various proof systems for classical logic which can be simply stated as follows:

Two proofs are the same iff they have the same combinatorial proof.

We will conclude by giving an overview how the combinatorial proof syntax can be extended and refined to express proofs for logics such as relevant [13, 2], modal [3], first order [11, 12], intuitionistic [8] and constructive modal [4] logic; the importance of this problem for mathematicians, computer scientists and philosophers [14]; as well as presenting the current challenges in extending this syntax to, e.g., fixed-point logics and higher order logics.

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Proof theory of Π^1_2-unsound theories

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The Π^1_2-soundness ordinal $o^1_2(T)$ of a theory T is a measure of how close T is to being Π^1_2-sound. We prove various results about the possible values of $o^1_2(T)$ and how these relate to the degree of soundness of T. For example, $o^1_2(T)$ is a recursive ordinal if and only if T proves a false Boolean combination of Π^1_1 sentences.

We also investigate the Π^1_2-Spectrum Conjecture, which asserts that the possible values of $o^1_2(T)$ for recursively enumerable extensions T of ACA_0 are precisely the Σ^1_1-definable epsilon numbers. We prove the following theorem:

Theorem Suppose that the Π^1_2-Spectrum Conjecture fails. Then, Second-Order Arithmetic is consistent.

If time allows, we might mention some results for Π^1_3 or Π^1_n.

First-Order Interpolation Derived from Propositional Interpolation[1]

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Ever since Craig's seminal paper on interpolation [3], interpolation properties have been recognized as important properties of logical systems. Recall that a logic L has *interpolation* if whenever $A \to B$ holds in L there exists a formula I in the common language of A and B such that $A \to I$ and $I \to B$ hold in L.

Propositional interpolation properties can be determined and classified with relative ease using the ground-breaking results of Maksimova cf. [4-6]. This approach is based on an algebraic analysis of the logic in question. In contrast first-order interpolation properties are notoriously hard to determine, even for logics where propositional interpolation is more or less obvious. For example it is unknown whether $\mathrm{G}^{\mathrm{QF}}_{[0,1]}$ (first-order infinitely-valued Gödel logic) interpolates (cf. [1]) and even for $\mathrm{MC}^{\mathrm{QF}}$, the logic of constant domain Kripke frames of three worlds with two top worlds (an extension of MC), interpolation proofs are very hard cf. Ono [8]. This situation is due to the lack of an adequate algebraization of non-classical first-order logics. In this paper we present a proof theoretic methodology to reduce first-order interpolation to propositional interpolation:

$$\left.\begin{array}{r}\text{existence of suitable skolemizations} +\\ \text{existence of Herbrand expansions} +\\ \text{propositional interpolation}\end{array}\right\} \Rightarrow \begin{array}{c}\text{first-order}\\ \text{interpolation.}\end{array}$$

The construction of the first-order interpolant from the propositional interpolant follows this procedure:

1. Develop a validity equivalent skolemization replacing all strong quantifiers[2] in the valid formula $A \to B$ to obtain the valid formula $A_1 \to B_1$.

[1]This abstract is based on the publication [2].

[2]Here we are dealing with quantifiers $\forall$ and $\exists$ such that $A(t) \to \exists x A(x)$ and $\forall x A(x) \to A(t)$ hold. This occurrence of quantifiers is called weak, the dual occurrence is called strong.

2. Construct a valid Herbrand expansion $A_2 \to B_2$ for $A_1 \to B_1$. Occurrences of $\exists x B(x)$ and $\forall x A(x)$ are replaced by suitable finite disjunctions $\bigvee B(t_i)$ and conjunctions $\bigwedge B(t_i)$, respectively.

3. Interpolate the propositionally valid formula $A_2 \to B_2$ with the propositional interpolant I^*: $A_2 \to I^*$ and $I^* \to B_2$ are propositionally valid.

4. Reintroduce weak quantifiers to obtain valid formulas $A_1 \to I^*$ and $I^* \to B_1$.

5. Eliminate all function symbols and constants not in the common language of A_1 and B_1 by introducing suitable quantifiers in I^* (note that no Skolem functions are in the common language, therefore they are eliminated). Let I be the result.

6. I is an interpolant for $A_1 \to B_1$. $A_1 \to I$ and $I \to B_1$ are skolemizations of $A \to I$ and $I \to B$. Therefore I is an interpolant of $A \to B$.

We apply this methodology to lattice based finitely-valued logics and the weak quantifier and subprenex fragments of infinitely-valued first-order Gödel logic.

Note that finitely-valued first-order logics admit variants of Maehara's Lemma and therefore interpolate if all truth values are quantifier free definable [7]. For logics where not all truth-values are represented by quantifier-free formulas this argument does not hold, which explains the necessity of different interpolation arguments for e.g. $\mathrm{MC}^{\mathrm{QF}}$. We provide a decision algorithm for the interpolation property for lattice based finitely-valued logics.

Most results in interpolation are concerned with the question whether a given logic interpolates but not with the more general question, to check the minimal extensions with that property. Our framework allows for the calculation of the relevant first-order extensions, which is given by the calculation of the relevant propositional extensions. For classical logic we show in this way that the fragment with $\top, \wedge, \vee, \forall, \exists, \to$ interpolates.

Propositional interpolation is easily demonstrated for MC, one of the seven intermediate logics which admit propositional interpolation [**?**]. Previous proofs for the interpolation of $\mathrm{MC}^{\mathrm{QF}}$, the first-order variant of MC, are quite involved, [**?**]. This interpolation result is a corollary of the main statement of this approach.

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Results and ideas on proof theory for interpretability logics

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The notion of formal interpretation arises in several areas of (meta)mathematics and there exist many variations on it.

For a modal analysis, one assumes a very general version of that concept: An interpretation of a theory T into a theory T' is just a structure preserving translation t such that if $\mathsf{T} \vdash A$ then $\mathsf{T}' \vdash t(A)$.

Interpretability logics arise as an extension of the language of provability logic by means of a binary modal operator $\rhd$ capturing the relation of (relative) interpretability between two arithmetical theories:[1] The propositional formula $A \rhd B$ is then intended as the modal counterpart of the arithmetical formula $Int_{\mathsf{T}}(\ulcorner A^* \urcorner, \ulcorner B^* \urcorner)$ – where $Int_{\mathsf{T}}(x, y)$ is the formal predicate for relative interpretability over T – expressing the fact that the arithmetical theory T extended by A^* interprets the arithmetical theory T extended by B^*, where $*$ is any arithmetical realisation for the modal language.

Their origins date back to Visser's [9,10], who axiomatised by the system $\mathbb{IL}$ the basic modal framework for interpretability. On top of it, several extensions can be defined.

A relational semantics – aka Veltman semantics – was presented first by de Jongh and Veltman's [3] for $\mathbb{IL}$. More complex modal completeness proofs for extensions were developed by the same authors in subsequent years, and further techniques were introduced to achieve analogous results since the beginning of 2000s by Joosten and several collaborators [1].[2]

On the arithmetical side, by tweaking Solovay's proof strategy for $\mathbb{GL}$, it is also possible to prove the *arithmetical* adequacy of some extensions of $\mathbb{IL}$ wrt *different* arithmetical theories. Indeed, a most intriguing aspect of interpretability logics is a definite sensitivity to the base arithmetic one is considering.

Nevertheless, there are many further open questions in the field. In the present context, it is worth noticing that very few is known about proof theory for interpretability logics.[3]

[1] Here we consider arithmetical theories satisfying the Hilbert-Bernays-Löb provability conditions.

[2] Subsystems of $\mathbb{IL}$ are discussed in Kurahashi and Okawa's [4].

[3] Sasaki's [7] gives a standard sequent calculus for $\mathbb{IL}$; Hakonemi and Joosten's [2] presents a labelled tableaux system for some extensions of $\mathbb{IL}$ based on standard Veltman semantics.

In this talk,[4] this gap in the proof-theoretic analysis of interpretability logics is partially filled by introducing a family $\mathsf{G3IL}^\star$ of labelled sequent calculi which covers in a natural way a wide range of modal systems for interpretability. Their design is based on the methodology of explicit internalisation by Negri and von Plato [5]: these new calculi internalise the hybrid models by Verbrugge, usually called generalised Veltman structures [8].

The main contribution of the work I am proposing consists then of the design of modular sequent systems satisfying main structural desiderata, namely: admissibility of contraction and weakening, invertibility of logical rules, and a cut-elimination algorithm. Besides these properties, adequacy results for $\mathsf{G3IL}^\star$ wrt the standard axiomatic and semantic presentations are easily established. Moreover, I would also discuss some ideas concerning bicompleteness of these systems and the current state of my quest for a (uniform) terminating proof-search strategy for the calculi presented here.

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PANDAFOREST: Proof analysis AND Automated deduction FOr REcursive STructures (Project Announcement)

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We introduce the community to our PANDAFOREST project, a joint project between Czechia and Austria that started in July 2022. The overall research goal of the project can be stated as follows:

> *There are interpretations of Herbrand's theorem extending its scope to formal number theory, these results are at the expense of analyticity, the most desirable property of Herbrand's Theorem. Given the rising importance of formal mathematics and inductive theorem proving in many areas of computer science, developing our understanding of the analyticity boundary is essential.*
>
> *We tackle these issues using a relatively novel formulation of induction as sequences of proofs, referred to as proof schemata. Proof schemata allow a recursive finite representation of many proof theoretically interesting objects as well as proof structures studied by the automated theorem proving community. Additionally, proof schemata provide the perfect framework to discuss analytic completeness of the method we plan to develop. This type of cyclic representation has been gaining traction over the past few years and will in all likelihood play an integral role in automated theorem proving and proof theory in years to come. However, unlike other approaches to cyclic proof theory, we focus on the extraction of a finite representation of the Herbrand information contained in formal proofs. The development of a computational proof-theoretic method for the extraction of Herbrand information for expressive inductive theories is our main objective. Furthermore, we hypothesize that developments in the proof theory of induction, using our chosen methodology (CERES style proof analysis), will lead to the development of more powerful inductive theorem provers.*

The first steps in this direction are outlined in "Schematic Refutations of Formula Schemata"[1] and 'CERES for first-order schemata"[2]. In particular, we plan the following to investigate the following research questions raised by this earlier work:

- What is the precise nature of the unification problem introduced in [1], does there exists a procedure deciding if two schematic terms are unifiable, and does this type of unification show up outside of proof-theoretic domains?
- Is the method introduced in [1] complete modulo a given *point transition system* (a concept introduced in [1]), and in what arithmetic setting is it complete?
- What is the precise relationship between our schematic formalism and the cyclic proof formalism?

Some of these questions already have partial answers, for example, there exist sufficient conditions for deciding if a pair of schematic terms are unifiable [3]. While this work addresses proof theoretic issues, the successful development of a complete method will result in a new variant of resolution that handles induction, thus contributing to automated reasoning as well.

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Non-uniform complexity via non-wellfounded proofs

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Non-wellfounded proof theory is the study of possibly infinite (but finitely branching) proof trees, where appropriate global correctness criteria guarantee logical consistency. This area originates (in its modern guise) in the context of the modal μ-calculus [7,5], serving as an alternative framework to manipulate least and greatest fixed points, and hence to model inductive and coinductive reasoning. Since then, non-wellfounded proofs have been widely investigated in many respects, such as predicate logic [2], arithmetic [8], and proofs-as-programs interpretations [6,4]. Special attention in these works is drawn to *cyclic* (or *regular*) proofs, i.e. non-wellfounded proofs with only finitely many distinct subproofs, comprising a natural notion of finite presentability in terms of (possibly cyclic) directed graphs.

The *Curry-Howard* reading of non-wellfounded proofs has revealed a deep connection between proof-theoretic properties and computational behaviours [6,4]. On the one hand, the typical correctness conditions ensuring consistency, called *progressiveness* (or *validity*) criteria, correspond to totality: functions computed by progressing proofs are always well-defined on all arguments. On the other hand, regularity has a natural counterpart in the notion of *uniformity*: circular proofs can be properly regarded as programs, i.e. as finite sets of instructions, thus having a 'computable' behaviour.

In a recent work [3], the authors extended these connections between non-wellfounded proof theory and computation to the realm of *computational complexity.* We introduced the proof systems CB and CNB capturing, respectively, the class of functions computable in polynomial time (**FP**) and the elementary functions (**FELEMENTARY**). These proof systems are defined by identifying global conditions on circular progressing proofs motivated by ideas from *Implicit Computational Complexity* (ICC). In particular, the system CB morally represents a cyclic proof theoretic formulation of B, i.e. Bellantoni and Cook's function algebra for *safe recursion* [1].

In this paper we investigate the computational interpretation of more general *non-wellfounded* proofs, where finite presentability is relaxed in order to model *non-uniform complexity.* In particular we consider the class **FP**/*poly* of functions computable in polynomial time by Turing machines with access to *polynomial advice.* Equivalently, **FP**/*poly* is the class of functions computed by families of polynomial-size circuits. Note, in particular, that such classes include *undecidable problems*, and so cannot be characterised by purely cyclic proof systems or usual function algebras, which typically have only computable functions.

We define the system nuB ('non-uniform B'), allowing a form of non-wellfoundedness somewhere between arbitrary non-wellfounded proofs and full regularity, and show that nuB duly characterises $\mathbf{FP}/poly$. The characterisation theorem for nuB relies on an adaption of the techniques in [3] to the current setting. This requires $\mathsf{B}(\mathbb{R})$, a relativisation of Bellantoni and Cook's function algebra B to the set of function oracles $\mathbb{R}$, deciding properties of string length. In particular, as a byproduct of our proof method, we show that $\mathsf{B}(\mathbb{R})$ captures $\mathbf{FP}/poly$, a folklore-style result that, as far as we know, has not yet appeared in the literature.

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Proof complexity of (nondeterministic) branching programs via games

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Proof complexity studies the size of proofs in different proof systems. The field was originally motivated by a seminal result due to Cook and Reckhow in [3]: *co***NP** = **NP** if and only if there is a propositional proof system that has polynomial-size proofs for each propositional tautology. The contrapositive statement has led to what is called 'Cook's program' for separating **P** and **NP**: find superpolynomial lower bounds for stronger and stronger proof systems until a general method is found.

Buss-Pudlák games. *Prover-Adversary games* for proof complexity were introduced in [5]. At a high level, these are two-player games in which one player, *Prover*, tries to 'prove' a formula Φ, while the other player, *Adversary*, 'pretends for as long as possible' that there Φ is falsifiable.

The game initialises with Prover asking Φ and Adversary answering 0. Then Prover asks other formulas and Adversary assigns Boolean values to them in turn. Prover *wins* if Adversary eventually gives answers that constitute a *simple contradiction*, i.e. contradict some row of a truth table.

Naturally this game is determined, and it turns out that Prover strategies winning in $O(d)$ rounds correspond precisely to Frege proofs of Φ of height $O(d)$. In this way we can say that Buss and Pudlák's game *corresponds* to Frege. This correspondence can be extended to other systems by modifying the class of queries that Prover can make, e.g. bounded-depth for bounded-depth Frege, or circuits for extended Frege.

Branching programs. A *(deterministic) branching program* (BP) is a (rooted) directed acyclic graph G with two distinguished *sink* nodes, 0 and 1. Each non-sink node v of G is labelled by a propositional variable and has two outgoing edges, one labelled by 0 and the other by 1. They are generalised to *non-deterministic branching programs* (NBPs) by allowing more than two outgoing edges for non-sink nodes. We say that an assignment α *satisfies* a NBP G if there is a path from the root to the sink 1 'consistent' with α.

BPs are conjectured to be exponentially more succinct than Boolean formulas, since they non-uniformly correspond to log-space (**L**), as opposed to $\mathbf{NC}^1$. NBPs correspond to nondeterministic log-space **NL** accordingly.

Recently, in work by Buss, Das and Knop [1], the proof systems eLDT and eLNDT were presented for reasoning with BPs and NBPs, respectively.

Work in progress. Inspired by Cook's ideas in [2] we define a variation of the Prover-Adversary game by allowing queries to be (Boolean combinations of) (N)BPs. Our main aim is to characterise fragments of the systems eLDT and eLNDT by imposing appropriate conditions on the queries.

Proving appropriate upper bounds for the depth of strategies in games for eLDT and eLNDT requires us to consider Boolean combinations of (N)BPs. This requires some low level but routine coding work for eLDT, but for eLNDT we must further prove a non-uniform variant of the Immerman-Szelepcsényi theorem, $\mathbf{NL} = co\mathbf{NL}$ [4, 6], bespoke to our setting. In fact, this work-in-progress aims to exploit the proof theoretic setting to work with a significant simplification of Immerman–Szelepcsényi. Namely, the subtle inductive counting aspect of the argument is devolved to the proof level rather than the formula level.

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Towards higher-order proof complexity

(work in progress)

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Background. One of the hallmarks of *proof complexity* is a deep correspondence between weak theories of arithmetic and propositional proof systems. This constitutes a logical version of classical *uniform-nonuniform correspondences* common in complexity theory, and at the same time a nonuniform version of *witnessing* arguments (or *deterministic-nondeterministic* correspondences) common in proof theory. Indeed, the advent of proof complexity has exposed a resulting factorisation of these correspondences, most famously in the case of *polynomial time*, as visualised in Figure 1. At least one goal of proof complexity is to extend such correspondences to other classes, systems and theories.

Motivation. We are interested in the gap between traditional *bounded arithmetic* theories of proof complexity (typically below exponential-time) and weak theories of arithmetic in proof theory (typically above elementary computation). A nonuniform version of elementary computation is naturally given by 'higher-order' Boolean logic, an extension of propositional logic by *abstraction* and *application* operations at all finite types. The corresponding Frege-style proof system is nothing more than Church's *simple type theory* (STT) [4], restricted to Boolean ground type. Our starting point is the observation that this system in fact bears correspondence with the arithmetic theory $I\Delta_0 + \exp$, a well-known theory of elementary computation. Our main aim is to refine this observation into a bona fide family of correspondences, in the sense of Figure 1, according to levels of the (alternating) elementary hierarchy.

Work-in-progress. We develop higher-order versions of Buss' classical theories that climb up the elementary hierarchy at the same time as analogous restrictions on cuts in STT, in particular by controlling type level in induction principles. Our goal is to establish modular characterisations for levels of the elementary hierarchy, and also intermediate alternating-time-hierarchies by controlling (higher-type) quantifier complexity. Along the way, we establish propositional systems corresponding to Buss' second-order theories U_2^i, V_2^i in the guise of fragments of 3^{rd} order Boolean logic.

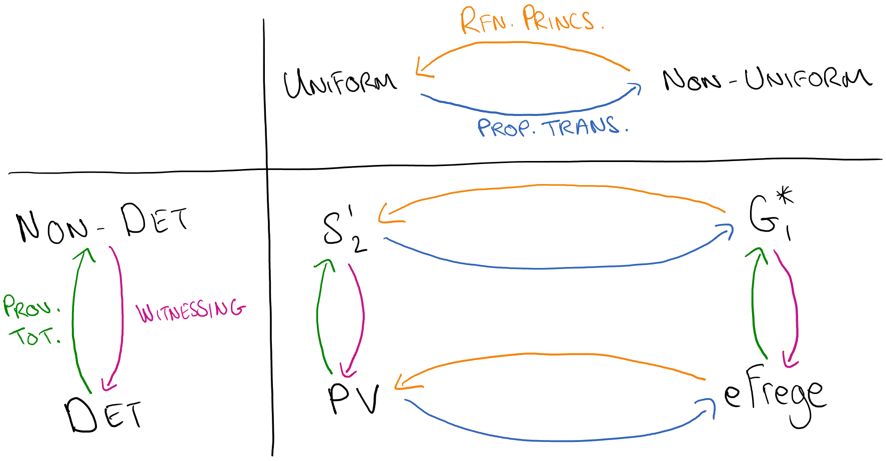

Figure 1: Proof complexity correspondences for *polynomial time*. The bottom row, the correspondence between PV and eFrege, is due to Cook in [1]. The left column, the correspondence between S^1_2 and PV, is due to Buss in [2]. The top row, the correspondence between S^1_2 and G^*_1, is due to Krajíček and Pudlák in [3]. The right column, the correspondence between G^*_1 and eFrege, is subsumed by the others, but may also be obtained independently (a folklore result).

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Intuitionistic versions of modal fixed point logics

(work in progress)

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Modal fixed point logics, such as the μ-calculus and PDL, have received increasing attention from proof theorists in recent years. Basic metalogical results for corresponding proof systems and axiomatisations are often obtained using an elegant mixture of techniques from automaton theory and proof theory, yielding a now healthy landscape of systems for reasoning about modal fixed point logics, e.g. [1,2,3].

On the proof theoretic side, however, the picture is far less clear. Treatments of cut-elimination, e.g. [4], and computational interpretations are far from robust and, furthermore, seemingly lack canonicity. At least one attempt to remedy this is to reduce the proof theory of fixed point logics to known well-behaved systems, for instance intuitionistic predicate logic, by means of double-negation and standard translations [5].

For comparison a similar approach can be carried out in pure modal logic without fixed points, constituting the proposal of 'Intuitionistic Modal Logic' IK of Fischer Servi and Plotkin & Stirling [6,7]. Indeed cut-elimination, and even normalisation for a natural deduction system, are obtained by Simpson in [8]. Characterised as the modal formulas whose standard translations are intuitionistically first-order provable, an interpretation of classical modal logic is duly obtained by composing with a specialised Gödel-Gentzen negative translation. On the other hand this feature of IK exposes significant differences between competing proposals for intuitionistic versions of modal logics [9], further exposing a lack of canonicity therein.

This work-in-progress aims to develop a bona fide theory of intuitionistic modal fixed point logic, from both axiomatic and semantic viewpoints, suitable for proof theoretic endeavours. Inspired by the success of IK and related logics, our guiding principle to this end is to characterise precisely the logic whose standard translations into the language of second-order logic (or of first-order logic with inductive definitions) are intuitionistically provable.

As a starting point, we aim to define a logic IK^+, whose simple fixed point language includes only two transitive closure modalities, $\Box^+$ and $\Diamond^+$. We can interpret these modal operators within predicate logic by way of different inductive definitions. We here report the ones for $\Diamond^+$:

$$
\begin{tikzcd}
\text{Hilbert systems} & \mathsf{IK}^+ \vdash A \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \mathsf{IFOLID} \vdash (A)^x \arrow[d, Leftrightarrow] \\
\text{Sequent calculi} & \mathsf{IK}^+ \vdash A \arrow[r, Rightarrow] & \mathsf{IFOLID} \vdash (A)^x \arrow[d, squiggly] \\
\text{Semantics} & \overset{?}{\models} A \arrow[u, squiggly] & \overset{?}{\models} (A)^x \arrow[l, squiggly]
\end{tikzcd}
$$

Figure 1: The blue arrows from $\mathsf{IK}^+ \vdash A$ to $\mathsf{IFOLID} \vdash (A)^x$ represent completed research. The red squiggly arrows represent a way of proving the converse direction, going through a class of birelational models that needs to be defined.

1. $(\Diamond^+ A)^x \iff \exists y(Rxy \wedge (A^y \vee (\Diamond^+ A)^y))$.
2. $(\Diamond^+ A)^x := \exists y(R^+ xy \wedge A^y)$, where $R^+ xy \iff Rxy \vee \exists z(Rxz \wedge R^+ zy)$.

These two definitions give rise to two different sets of inductive first order axioms, as well as (modal) sequent calculus rules within the framework of [10]. We prove the equivalence of conditions 1 and 2 in intuitionistic first order logic with inductive definitions (IFOLID). As a result, we are able to conclude that a natural modal axiomatisation of IK^+, obtained by extending IK by the fixed point characterisation $\Diamond^+ A \iff \Diamond A \vee \Diamond\Diamond^+ A$ (and similarly for $\Box^+$), is indeed interpreted into IFOLID; this result is indicated by the blue arrows in Figure 1.

In order to obtain the converse direction, we are currently investigating a semantics that validates the aforementioned standard translation, and for which IK^+ may be complete; these are the red squiggly arrows in Figure 1. Indeed, it is currently unclear whether natural frame conditions for $\Diamond^+$ in the birelational semantics for intuitionistic modal logic are approriate to this end.

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Natural deduction derived from truth tables

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In this talk we explore a general method for deriving natural deduction rules from truth tables developed in [3,4] which we call *truth table natural deduction*. Arbitrary connectives defined via a truth table give rise to introduction and elimination rules. The method applies to both classical and intuitionistic propositional logic. This implies that we can derive 'intuitionistic valid' rules for any (classical) connective. The rules in this method adhere to a standard format, which gives a general view on well-known natural deduction systems.

In this talk we would like to focus on the intuitionistic truth table system and its properties. In short, we will investigate the definition of the rules, Kripke semantics, and a Curry-Howard correspondence including strong normalization of proof reductions. If time permits, we would like to comment on ongoing research of the classical system.

The truth table natural deduction system defines inference rules for a set of connectives, where each connective has a truth table. The method applies to connectives with arbitrary arity, such as the connective if-then-else of arity 3. Each row $b_1 \dots b_n | b$ of the truth table of an n-ary connective gives rise to an elimination rule in case $b = 0$ or an introduction rule in case $b = 1$. The form of the rule is determined by the assignments of $b_1, \dots, b_n$. In this way, each n-ary connective has 2^n inference rules. For example, $\wedge$ has three elimination rules and one introduction rule. Using optimalization lemmas, the number of rules for a connective can be reduced.

We construct a general Kripke semantics which we show to be sound and complete for the intuitionistic truth table natural deduction rules. This justifies the constructive flavour of the rules. The Kripke semantics allows us to easily prove a generalization of the disjunction property.

Proof reductions are important in natural deduction systems, because they give insight in the interaction between the introduction and elimination rules. The question is whether each derivation can be transformed into a normal derivation in which no 'superfluous' inference rules are used. The standard format of the truth table system gives rise to general notions of detour and permutation conversion. Weak normalization of the reductions implies properties such as consistency, the subformula property, and decidability [3,4].

In [2], we have proved that reduction in the intuitionistic truth table system is strongly normalizing. Strong normalization is difficult to prove, because reduction of the truth table natural deductions is non-deterministic. This makes it challenging to study and the normal forms are not unique. To prove strong normalization, we generalize a method from [5]. We construct a conversion-preserving translation from deductions to terms in an extension of simply typed lambda calculus which we call *parallel simply typed lambda calculus* and we prove that it is strongly normalizing.

The connection between our system and different natural deduction systems makes it interesting to study. For the well-known connectives $\vee, \wedge, \to$ and $\neg$, the rules in the truth table natural deduction system are equivalent to the natural deduction rules from Gentzen and Prawitz [1,8]. However, they are in a different shape. Other related research is done on the generalized elimination rules [7] and deriving classical introduction rules from truth tables [6].

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Challenges to Instrumentalism

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The axiomatic method can be seen as an extrapolation of knowledge that is obtained in an almost immediate fashion. For example, in the realm of the natural numbers, the axiom $a + b = b + a$ or rather its universal closure can be seen as an extrapolation of nearly sensorial knowledge that joining two quantities is independent on their order. Hilbert would speak of statements of *finitistic* nature.

Via Gödel's famous incompleteness theorems we know that any reasonable axiomatic theory T is incomplete in that certain true statements will not be provable within T. Expanding our knowledge beyond the limits of T thus requires new axioms that mostly can be regarded as *extra-sensorial* or *extra-empirical* as opposed to $a+b = b+a$. This leaves a large justificational burden on which extra-empirical principles to accept and which not.

The school of *Instrumentalism* implies a relaxed epistemology and the question whether a new principle/axiom should be accepted and regarded as true is replaced by a merely instrumental viewpoint: if the new principle is *useful* for explaining and predicting real-life phenomenon then one should adopt it.

In this talk we present some discussions and work in progress where we consider how various results on speed-up of proofs can be interpreted as support for or challenges to Instrumentalism.

As such we shall be testing the thesis of Instrumentalism using various scenarios that model possible situations an instrumentalist could see herself be confronted with. In particular, we will present situations where adding both a sentence and its negation will yield speed-up. Which choice is the intstrumentalist to make in such a case. In a sense, our approach thus has a flavour similar to Solovay's challenge to Nelson's thesis on Predicate Analysis. Nelson proposed in 1986 that Predicative Analysis should be called that part of mathematics that can be interpreted in Robison's arithmetic Q. Solovay provided a serious challenge to this program, to not say that he ended it, by providing an Orey sentence for Q: a sentence so that itself but also its negation can be interpreted in Q.

In our situation is not so clear since what accounts for substantial criticism since there is no clear formalised claim as to what instrumentalism should embody. As a matter of fact, Instrumentalism is often mentioned in the realm of theories for Physics. There, it says that physical theories do not necessarily need to give an ontological account of their theoretical artefacts. If an artefact like an unobservable quark is useful, then that is its justification and ontological questions about the artefact itself may lack to have a particular truth value.

There seems to be no general consensus what *usefulness* is understood to mean. One can think of how an artefact gives rise to shorter proofs, more elegant proofs, or if it helps unifying fields that were unrelated before, or if it brings more uniformity in the various fields. Notions of elegance or ability of unification are difficult to pin down, and in this presentation we shall merely focus on the usefulness with respect to providing shorter proofs. Having settled on this restrictive modelling assumption we can now see how existing and new results on speed-up can be used to test positions related to Instrumentalism.

A new proof system for the modal μ-calculus inspired by determinisation of automata

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We define an annotated cyclic proof system for the modal μ-calculus using a determinisation method for nondeterministic parity automata. This proof system is then embedded into the cyclic system Clo presented in [1]. As a corollary we obtain a relatively straight-forward completeness proof of Kozen's axiomatisation of the μ-calculus.

In [4] tableaux games for the modal μ-calculus were introduced, in which the first player has a winning strategy iff the formula is satisfiable. The winning condition can be naturally checked by a nondeterministic parity automaton. The determinisation of that automaton may be used to obtain a proof system, which has already been accomplished by Jungteerapanich [3], motivated by the Safra construction [5]. We define a determinisation method for nondeterministic parity automata, called the binary tree construction. This method adds a different perspective and new ideas to concepts introduced in [2]. Using the binary tree construction we define the annotated cyclic proof system BT, where the annotations are tuples of binary strings. Moreover, we make explicit use of the deterministic automaton to obtain soundness and completeness of the proof system.

The main benefit of our proof system is, that all the information of the automaton is stored in the annotations and no extra control as in [3] is needed. On the downside, the soundness condition has to speak about every strongly connected component of the cyclic proof tree. Nevertheless, we can adapt the notion of a monotone proof from [1], which circumvents this issue. We show that every BT-proof can be transformed to a monotone one along the lines of [6]. Every monotone BT-proof can be translated to a proof in the cyclic proof system Clo presented by Afshari, Leigh [1]. As Clo can be embedded into Kozen's axiomatisation Koz of the μ-calculus we obtain a simplification of the completeness proof of Koz in [1].

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A Cyclic System for Arithmetical Inductive Definitions

(work in progress)

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In recent years we have seen increasing interest in cyclic proof systems. These are systems where proofs can be non-wellfounded, i.e. contain infinite branches, but nonetheless have only finitely many distinct sub-proofs. This non-wellfoundedness allows them to simulate inductive arguments without need for explicit induction axioms. Naturally some appropriate correctness criterion is required to prevent fallacious reasoning, usually in the form of some ω-regular property of infinite branches (the *trace condition*).

In this work we develop a cyclic version of the theory ID_1, called CID_1. ID_1 is a well-known arithmetic theory that extends PA by arithmetical inductive definitions. CID_1 is a cyclic version of ID_1 similar to how the theory of cyclic arithmetic (CA) was developed as a cyclic version of PA by Simpson [1]. The system is similar in spirit to Brotherston's cyclic system for ordinary inductive definitions over first-order logic [2].

Our main results are the soundness of CID_1 (with respect to the standard model) and that ID_1 and CID_1 prove the same arithmetical sentences. Both results are proved similarly to the case of CA and PA in [1] at a high level, but provide significant new technical challenges at the low level.

To prove soundness we assume the existence of a proof of a false statement and then construct an infinite false branch that, under the trace condition, yields a contradiction. In contrast to the case of CA infinite 'false branches' are not completely ruled out by the trace condition and so we have to be more careful to choose a branch with the appropriate properties. This is similar in complexity to the countermodel branch construction required for logics with more complex fixed pionts, such as the μ-calculus, cf. [3].

To show conservativity we employ a metamathematical argument, formalising the soundness of CID_1 within (a theory conservative over) ID_1 and then appeal to *reflection*. The big challenge for this approach, as compared to the similar argument for CA, is that the closure ordinals of our inductive definitions (up to Church-Kleene) far exceed the proof theoretic ordinal of the theory (Bachmann-Howard) and so explicit induction on their notations is not possible for the soundness argument. Thus a different approach is required, namely requiring a formalisation of the *theory* of (recursive) ordinals at the object level. We expect that similar results carry over to $\mathsf{ID_n}$, the theory of n (nestings of) inductive definitions, for each $n \in \omega$ and thus $\mathsf{ID}_{<\omega}$, but this is currently work-in-progress.

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Looking inside the modalities: subatomic proof theory for modal logics

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Deep inference [1,6] is a proof formalism allowing to rewrite formulas deep inside an arbitrary context. This gives more flexibility in the design of inference rules, and has also been used to design proof systems for modal logics [5,7]. However, cut-elimination for deep inference systems is more involved than for traditional sequent style systems. In particular, for modal logics, no cut-elimination proof that is internal to deep inference has been given so far.

There are many different cut-elimination techniques in the deep inference literature, exploiting different aspects of the proof systems they work on. A particular methodology does however stand out for its generality: *cut-elimination via splitting* [2]. Even though this proof has to be redone for every proof system anew, there is a certain repeating pattern. In order to formalize this pattern and to obtain a general cut-elimination method that works for many proof systems at the same time, the method of *subatomic proof theory* [4] has been developed. The basic idea is to treat atoms as binary connectives, leading to a uniform shape of all inference rules. This enormously reduces the number of cases in the case analysis for cut-elimination. A proof of cut-elimination via splitting usually consists of two parts. Only the second one is the actual splitting and needs a "linear" system, i.e., one without weakening and contraction. To remove weakening and contraction, the first part of the cut-elimination performs a *decomposition* [2] or *cycle elimination* [3].

In this work in progress we present a subatomic proof system for classical modal logic called $\mathsf{SAK^KS}$ which is sound and complete for the modal logic K (presented in Figure 1). Since the subatomic methodology treats atoms like binary connectives we extend this phenomenon to also treat the unary modalities as binary connectives.

The set of subatomic formulae for classical modal logic is given by the set of constants $\mathcal{U} = \{\mathsf{0}, \mathsf{1}\}$ and the set of connectives $\mathcal{R} = \{\boxtimes, \wedge, \vee\} \cup \mathcal{A}$ where $\mathcal{A}$ is a countable set of atoms, denoted by $a, b, ...$ with $\mathcal{A} \cap \{\boxtimes, \wedge, \vee\} = \emptyset$. The intuitive idea is to interpret $\mathsf{0}a\mathsf{1}$ as a positive occurrence of the atom a, and $\mathsf{1}a\mathsf{0}$ as a negative occurrence $\overline{a}$ of the same atom. The operator $\boxtimes$ allows us to capture the modalities. An interpretation function $\mathtt{I} : \mathcal{F} \to \mathcal{G}$ is a partial

$$
\mathsf{sai}_{\downarrow}\ \frac{(A \vee B)\ a\ (C \vee D)}{(A\ a\ C) \vee (B\ a\ D)} \qquad
\mathsf{sai}_{\uparrow}\ \frac{(A\ a\ B) \wedge (C\ a\ D)}{(A \wedge C)\ a\ (B \wedge D)}
$$

$$
\wedge_{\downarrow}\ \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad
\wedge_{\uparrow}\ \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}
$$

$$
\mathsf{m}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C)\ a\ (B \vee D)}
$$

$$
\mathsf{sac}_{\downarrow}\ \frac{(A\ a\ B) \vee (C\ a\ D)}{(A \vee C)\ a\ (B \vee D)} \qquad
\mathsf{sac}_{\uparrow}\ \frac{(A \wedge B)\ a\ (C \wedge D)}{(A\ a\ C) \wedge (B\ a\ D)}
$$

$$
\mathsf{sak}_{\downarrow}\ \frac{(A \vee C) \boxtimes (B \vee D)}{(A \boxtimes B) \vee (C \boxtimes D)} \qquad
\mathsf{sak}_{\uparrow}\ \frac{(A \boxtimes C) \wedge (B \boxtimes D)}{(A \wedge B) \boxtimes (C \wedge D)}
$$

Figure 1: System $\mathsf{SAK^KS}$

function where $\mathcal{F}$ is the set of subatomic formulae and $\mathcal{G}$ the set of formulae of classical modal logic and it is defined as follows:

$$
\begin{array}{lcc}
\mathtt{I}(\mathsf{1}) = \mathsf{1} & \forall a \in A.\ \mathtt{I}(\mathsf{0}a\mathsf{0}) = \mathsf{0} & \forall a \in A.\ \mathtt{I}(\mathsf{1}a\mathsf{1}) = \mathsf{1} \\
\mathtt{I}(\mathsf{0}) = \mathsf{0} & \forall a \in A.\ \mathtt{I}(\mathsf{0}a\mathsf{1}) = a & \forall a \in A.\ \mathtt{I}(\mathsf{1}a\mathsf{0}) = \overline{a} \\
 & \mathtt{I}(A \vee B) = \mathtt{I}(A) \vee \mathtt{I}(B) & \mathtt{I}(A \wedge B) = \mathtt{I}(A) \wedge \mathtt{I}(B) \\
 & \mathtt{I}(\mathsf{1} \boxtimes A) = \Box A & \mathtt{I}(\mathsf{0} \boxtimes A) = \Diamond A
\end{array}
$$

where A, B are interpretable formulae. An example of subatomic formulae for modal logic is $A \equiv ((\mathsf{0} \boxtimes A) \vee (\mathsf{0}a\mathsf{1})) \wedge (\mathsf{1} \boxtimes A)$ and its interpretation is $\mathtt{I}(A) = (\Diamond A \vee a) \wedge \Box A$.

We can show that for every interpretable $\mathsf{SAK^KS}$ derivation with premiss P and conclusion C, there is a derivation in the normal deep inference system for modal logic called $\mathsf{SKS\text{-}K}$ [7] with premiss $\mathtt{I}(P)$ and conclusion $\mathtt{I}(C)$. In other words, we have the following:

Theorem 1 *The system* $\mathsf{SAK^KS}$ *is conservative extension for the deep inference system* $\mathsf{SKS\text{-}K}$*.*

Corollary 2 *The system* $\mathsf{SAK^KS}$ *is sound and complete with respect to the modal logic* K*.*

Furthermore, we show cut-elimination via splitting for the linear fragment of the system.

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A Defense of Proof-Theoretic Logical Pluralism

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Restall (2014) presents a defense of a meaning-invariant logical pluralism from a proof-theoretic perspective. In short, meaning-invariant logical pluralism is the joint assertion of the following two theses: i) there is more than one correct logic; and ii) these correct logics share one and the same language (Restall, 2014, p. 281). A central point in this position, then, is that the logical connectives have the same meaning across the different logics at issue.

Restall works with the case of classical, intuitionistic, and dual intuitionistic logic. In Restall's view, language preservation is achieved across these three logics because their connectives are governed by the same inference rules in the standard presentations of their respective sequent calculi.

Dicher (2016) has pointed out that Restall's view assumes a particular criterion of rule individuation, which he dubs the sameness claim: inference rules that differ only with respect to the structures of their derivability relations are identical. According to Dicher this thesis is false, since it entails an incorrect view of what inference rules are.

In this talk, my aim is to show that Dicher's argument against the sameness claim is unsound. In particular, his understanding of the structure of the derivability relation does not fit the notion of 'structure' at play in Restall's proposal.

I proceed in two stages. First, I appeal to Mares' and Paoli's (2014) renewal of Avron's (1988) distinction between internal and external consequence relation and argue that in Restall's proposal, the sameness claim should be understood as holding at the level of the internal consequence relation. Then, I argue that the particular rules used by Dicher to disprove the sameness claim do not differ only with respect to the structures of the internal consequence relations of the calculi they are used in. Based on this, I claim that Dicher has not showned that the sameness claim is false.

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A computational study of a class of recursive inequalities

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The proof mining program aims to give a computational interpretation to prima facie non-effective proofs through the application of tools from logic. In recent years, proof mining has enjoyed many successes in nonlinear analysis, with logical tools being used to extract very uniform bounds (e.g. bounds independent of the space). In this work, we present a contribution to the proof mining of nonlinear analysis.

Recursive inequalities play a big role in nonlinear analysis. A common way they are used is in establishing the convergence of an iteratively defined sequence of elements in some space to a point satisfying some properties. A simple example of this can be seen through the Banach fixed point theorem where, it can be shown that, for a contraction mapping T with constant $c \in [0, 1)$ and x^* a fixed point of T, the distance $\mu_n := d(T^n x_0, x^*)$ satisfies $\mu_{n+1} \leq c\mu_n$ and thus converges to 0.

In our work, we study the convergence properties sequences of nonnegative real numbers $\{\mu_n\}$ and $\{\beta_n\}$ satisfying,

$$\mu_{n+1} \leq \mu_n - \alpha_n \beta_n + \gamma_n \tag{1}$$

with $\{\alpha_n\}$ a nonnegative sequence of real numbers with a divergent sum and $\{\gamma_n\}$ a nonnegative sequence of real numbers that converges to 0. This recursive inequality features in numerous optimization problems in nonlinear analysis. Typically α_n represents some step size for an algorithm and γ_n represents an error term.

One can easily produce examples where the condition that $\gamma_n \to 0$ is not enough to deduce the convergence of either $\{\mu_n\}$ or $\{\beta_n\}$. Thus, in the literature this condition is usually strengthened to one of the two cases:

(I) $\sum_{i=0}^{\infty} \gamma_i < \infty$

(II) $\gamma_n/\alpha_n \to 0$ as $n \to \infty$.

We study each of these cases in turn and obtain quantitative results about the convergence of $\{\mu_n\}$ and $\{\beta_n\}$ by producing computable rates of convergences, in some cases.

It is a known result of Specker [1] that it is not always possible to obtain a computable rate of convergence for converging sequences of computable numbers. In our work we also produce similar negative results. In scenarios where it is impossible to produce a computable rate of convergence we obtain, instead, a rate of metastablity. This is a functional $\Phi : \mathbb{Q}_+ \times (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ satisfying,

$$\forall \varepsilon \in \mathbb{Q}_+ \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists n \leq \Phi(\varepsilon, g) \, \forall k \in [n, n + g(n)](|a_k - a| \leq \varepsilon) \tag{2}$$

where, $[a, a + b] := \{a, a + 1, \ldots, a + b\}$.

The idea of metastability comes from logic. If one takes takes the Herbrand normal form of the definition of convergence, we obtain a finitary version of this principle (in the sense of Tao [8]). A rate of metastability will be a computable interpretation to this definition and can be recognised as being a solution to the so-called 'no-counterexample interpretation' of the definition of convergence [2,3]. Obtaining rates of metastability using proof theoretic techniques is a standard result in applied proof theory (e.g.[4,5,6]).

After an abstract study of recursive inequalities, we discuss how our results about the convergence properties of real numbers have application in nonlinear analysis. We adapt the work of Alber et al. in [7], to produce a general gradient descent algorithm and rates of metastability for the convergence of our algorithm to a solution. Furthermore, we are able to pin point the exact ineffective principles which stopped the authors of [7] from being able to produce explicit rates of convergences for their algorithm. In addition, we demonstrate how our work generalises known results in the proof mining literature such as the study of Mann schemes for asymptotically weakly contractive mappings [9] and in the study of set values accretive operators ([10] for example).

Alongside this theoretical work, we have also started a Lean library [1] devoted to implementing quantitative results that use recursive inequalities. This work will be useful as it would allow us to have implemented a large class of core lemmas used in both in the formalization of nonlinear analysis and proof theoretic applications. Our formalization project is still very much in its early stages, with only a handful of known rates of convergences and metastabilities from the literature, to date, being verified. In addition, we have also implemented a key construction, from computable analysis, of a sequence of rational numbers converging to zero without a computable rate of convergence. This sequence has been adapted allover the applied

[1] https://github.com/mneri123/Proof-mining-

proof theory literature to produce negative results, of the type previously discussed. I shall discuss interesting aspects of the formalization that has been done so far and also outline future directions for research in both implementation and potentially automated reasoning.

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Π^1_2 proof-theoretic analysis of well-ordering principles

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Dilators are a certain kind of operators on linear orders that map well-orders to well-orders. Many natural operators on well-orders could be naturally extended to dilators, for example $X \mapsto X2$, $X \mapsto \omega^X$, $X \mapsto \varepsilon_X$. Formally, a pre-dilator D is a continuous stable autofunctor on the category of linear orders and strictly increasing maps. A pre-dilator is a dilator if it maps well-orders to well-orders.

By Girard's normal form theorem each pre-dilator is naturally isomorphic to a denotation systems D. Which is a special kind of functor given by a term system consisting of terms $t(x_1, \ldots, x_{n_t})$ and a system of comparison rules between terms, where the comparison between two terms is determined purely by the pairwise comparisons between their arguments. Each $D(A)$ consists of all the terms $t(a_1, \ldots, a_{n_t})$, $a_1 <_X \ldots <_X a_n$ compared according to the rules. For $f\colon A \to B$ we put $D(f)(t(a_1, \ldots, a_{n_t})) = t(f(a_1), \ldots, f(a_{n_t}))$.

The notion of a countable denotation system could be easily formalized in the language of second-order arithmetic. When we will be talking about dilators/pre-dilators in systems of second-order arithmetic we will be always talking about functors that are denotations systems.

We aim to study provable computable dilators of various systems of second-order arithmetic. That is, for a given system of second-order arithmetic T we are interested in the set of all provable computable dilators, i.e. computable denotation systems D such that T proves that D maps well-orders to well-orders. Since the set of all indexes of computable dilators is Π^1_2-complete, in fact a characterization of provable computable dilators of T is a characterization of the Π^1_2-consequences of T.

Let $\lambda X.\omega^X$ and $\lambda X.X + 1$ be the naturally defined denotation systems. Let $\lambda X.\omega_n^{X+1}$ be $(\lambda X.\omega^X)^n \circ (\lambda X.X + 1)$.

The following theorem is constitutes Π^1_2 proof-theoretic analysis of ACA_0.

Theorem 1. *If D is ACA_0-provable computable dilator, then for some n, there is a natural transformation $\eta\colon D \to \lambda X.\omega_n^{X+1}$.*

We generalize Theorem 1 to a Π^1_2 proof-theoretic analysis of extensions of ACA_0 by the principles $\mathsf{Dil}(F)$ asserting that a computable dilator D indeed is a dilator, i.e. the well-ordering principles.

Theorem 2. *Suppose D is a computable dilator. Then if D is $(\mathsf{ACA}_0 + \mathsf{Dil}(D))$-provable computable dilator, then for some n, there is a natural transformation $\eta\colon D \to (\lambda X.\omega^X + F(X) + 1)^n$.*

Provability games for Modal Logics

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The idea of using the game approach in logic is not new and has a long tradition [1]. There are multiple ways in which games can enter the realm of logic. Games can be either an object of study, e.g., in game logic, or rather used as a tool to analyse and deal with various logical systems. Games as tools can be used as proofs or semantics for various logics. There is a great variety of different logic games designed for various tasks. In this project, however, we are concerned only with two major types of logical games, namely semantic games and provability games. Semantic games are played over a particular model $\mathcal{M}$ and aimed at identifying truth in $\mathcal{M}$, i.e., whether a given formula, or a set of formulae, is true in $\mathcal{M}$. The most well-known example is Hintikka's *game-theoretical semantics* (GTS) [2]. Provability games, on the contrary, correspond to the notion of validity or, more generally, the relation of logical consequence with the most prominent example being *dialogue logic* of Lorenzen and Lorenz [3].

We present *Mezhirov game*, an alternative provability game initially proposed by Iliya Mezhirov for intuitionistic logic **IPC** as well as *Grzegorczyk modal logic Grz* [4]. Among the significant features of the game are its finiteness and explicit reference to truth values. It is a two-player zero-sum game that, though being the game on validity, is closely related to Kripke semantics. The original approach is extended here to modal logics. The main results are new games that not only are finite but also shed light on the relation between the semantic and the syntactic approaches to validity.

The idea is very simple: the Proponent **P** tries to prove that the formula is valid whereas the Opponent **O** seeks to show that this does not hold, i.e., that there exists a model and interpretation such that the formula in question is not satisfied there and, hence, not valid. For the simplicity reasons, we describe a game for Johansson Minimal Logic. The game $\mathcal{G}(\varphi)$ played over the formula φ contains the following elements:

- φ is the initial formula of the game.
- $\mathcal{F}$ is the set of all subformulae of the initial formula φ.
- the set of players $\mathcal{A} = \{\mathbf{O}, \mathbf{P}\}$ where $\mathbf{P}$ is *Proponent* and $\mathbf{O}$ is *Opponent*;

- players' corresponding sets of moves will be denoted by $\mathcal{P}$ and $\mathcal{O}$.
- a position, or a game state, C is a pair $(\mathcal{P}, \mathcal{O})$, where $\mathcal{O}$ and $\mathcal{P}$ are sets of subformulae of φ. The number of possible positions is finite, since there number of formulae is finite. The starting position is $C_0 = (\varphi, \emptyset)$.
- game valuation function $v : \mathcal{F} \times \mathcal{C} \longrightarrow \{0, 1\}$ defined recursively. Game valuation of each subformula of φ is calculated for every position in the game, hence we denote valuations in particular games as v_i read as the valuation of ψ at some game state i. $\mathcal{V}$ is the set of all possible valuations. For all game position C and all $\psi_i, \psi_j \in \mathcal{F}$, $p \in Prop$, $\star \in \{\wedge, \vee, \rightarrow\}$:

$$v(\bot, C) = 1 \text{ iff } \bot \in \mathcal{O} \tag{1}$$

$$v(p, C) = 1 \;\text{ iff }\; p \in \mathcal{O} \tag{2}$$

$$v(\psi_i \star \psi_j, C) = 1 \Longleftrightarrow \begin{cases} (\psi_i \star \psi_j) \in (\mathcal{O} \cup \mathcal{P}) \\ v(\psi_i) \star_B v(\psi_j) = 1^1 \end{cases} \tag{3}$$

where $\star_B$ is the Boolean function associated with $\star$. The curly bracket should be read as a conjunction.

The last condition can be read as follows: a formula that is not in $\mathcal{O} \cup \mathcal{P}$ (also called a non-marked) is always false, and a marked formula behaves according to its classical truth table.

The game proceeds as follows:

* Players *move* by adding a formula to their respective sets ($\mathcal{P}$ and $\mathcal{O}$) which is called *marking* a formula.
* If a player has marked a formula and its game valuation is 0, let us say that this formula is his **mistake**. If **O** has no mistakes but **P** has, then it's **P**'s turn to move. Otherwise, it's **O**'s turn.
* The game terminates when a player whose turn it is cannot move. A player *loses* when he cannot move.

The game is tightly related both to the Kripke semantics as well as the proof theory since the winning strategies for **P** can be seem as derivations in a sequent calculus and countermodel can be extracted from **O**'s winning strategies. In the talk we are going to present games for several Modal Logics as well as their relation to Kripke semantics and existing proof systems.

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The proof-theoretic square

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Prawitz semantics of valid arguments (SVA) [3] is a well-known example of proof-theoretic semantics. It can be understood as an extension of Prawitz's own normalisation theory for Gentzen's natural deduction [4], based on what Schroeder-Heister [5] called the "fundamental corollary" of Prawitz's normalisation: in certain (important) systems, closed normal derivations end by introduction. This can be taken as a sort of "semantic confirmation" of Gentzen's claim that introductions fix the meaning of the logical constants, whereas eliminations are unique functions of the introductions [2]. Following Dummett's [1] principle that, if A is provable, then A must also be canonically provable (where "canonical" means "ending by introduction"), one can thus simply turn the corollary into a semantic requirement, and replace derivations with arbitrary argument-structures $\mathscr{D}$, whose non-introductory inferences be justified by equally arbitrary justification procedures $\mathfrak{J}$. The general idea of SVA is framed by the case where $\mathscr{D}$ is closed, i.e. has no free variables or assumptions: $\langle \mathscr{D}, \mathfrak{J} \rangle$ is valid iff, by applying in some order the elements of $\mathfrak{J}$, $\mathscr{D}$ reduces to a canonical form whose immediate sub-structures are valid when paired with $\mathfrak{J}$.

However, this general idea must be refined through some semantically crucial intuitions. Firstly, arguments may be simply *locally* valid, e.g. because of some specific meaning of the non-logical terminology they involve; if not, the argument can be said to be *logically* valid. Hence, we have to specify how non-logical meaning is determined, but in doing this, we must comply with a second fact, namely, that argument-structures may be open, so that validity should be defined for the open case too.

In SVA, determination of non-logical meaning is achieved through *atomic systems* Σ, i.e. the meaning of the non-logical terminology is given in terms of deductive use of this terminology in purely atomic derivations. Local validity becomes validity over an atomic system, and validity in the open-case is dealt with through a kind of closure principle, i.e. open arguments are valid when all their closed instances are so. At this point, SVA faces with a first dilemma: when closing open arguments, should we require validity over one and the same system, or should we require that the property is preserved over extensions of the system?

Definition 1 $\langle \mathscr{D}[x_1, .., x_n, A_1, ..., A_m], \mathfrak{J} \rangle$ *is* NE*-valid over* Σ *iff, for every*

k_i in the language of Σ, for every closed $\langle \mathscr{D}_j, \mathfrak{J}_j \rangle$ for A_j valid over Σ, $\langle \mathscr{D}[k_1, ..., k_n, \mathscr{D}_1, ..., \mathscr{D}_m], \mathfrak{J} \cup \mathfrak{J}_1 \cup ... \cup \mathfrak{J}_m \rangle$ is valid over Σ.

Definition 2 *$\langle \mathscr{D}[x_1, .., x_n, A_1, ..., A_m], \mathfrak{J} \rangle$ is `WE`-valid over Σ iff, for every Σ^+, for every k_i in the language of Σ^+, for every closed $\langle \mathscr{D}_j, \mathfrak{J}_j \rangle$ for A_j valid over Σ^+, $\langle \mathscr{D}[k_1, ..., k_n, \mathscr{D}_1, ..., \mathscr{D}_m], \mathfrak{J} \cup \mathfrak{J}_1 \cup ... \cup \mathfrak{J}_m \rangle$ is valid over Σ^+.*

This distinction is anything but trivial for, as shown by Schroeder-Heister [5], it determines whether the overall semantics is or not monotonic.

Proposition 3 *`NE`-validity is non-monotonic, i.e. there is $\langle \mathscr{D}, \mathfrak{J} \rangle$ such that, for some Σ, for some Σ^+, $\langle \mathscr{D}, \mathfrak{J} \rangle$ is `NE`-valid over Σ and $\langle \mathscr{D}, \mathfrak{J} \rangle$ is not `NE`-valid over Σ^+.*

Proposition 4 *`WE`-validity is monotonic, i.e. for every $\langle \mathscr{D}, \mathfrak{J} \rangle$, for every Σ, if $\langle \mathscr{D}, \mathfrak{J} \rangle$ is `WE`-valid over Σ then, for every Σ^+, $\langle \mathscr{D}, \mathfrak{J} \rangle$ is `WE`-valid over Σ^+.*

Concerning logical validity, however, we have a second dilemma, for there seem to be at least two SVA compatible ways for understanding independence from non-logical meaning. This stems from the fact that arguments are *pairs* consisting of an argument-structure *plus* a set of justification procedures. Quantification on non-logical meanings must be hence accompanied by quantification on sets of justification procedures, and this returns again two alternatives.

Definition 5 *$\Gamma \models_1 A$ iff there is $\mathscr{D}$ with assumptions Γ and conclusion A such that, for every Σ, for some $\mathfrak{J}$, $\langle \mathscr{D}, \mathfrak{J} \rangle$ is valid over Σ.*

Definition 6 *$\Gamma \models_2 A$ iff $\Gamma \models_1 A$ with respect to a fixed set of justifying functions, the validity of which (with respect to their type) be provable.*

My aim in this talk is twofold. First, I argue that the options in the alternatives above have advantages and shortcomings which are symmetric both from an internal, and from an external point of view - i.e. both within the same alternative, and with respect to the other one.

`NE` has the shortcoming of returning a non-monotonic notion of validity, which is not in line with the idea that if an argument is valid, it should remain so when expanding the knowledge base. But `NE` has also the advantage of accounting for the very natural idea that arguments may be valid, not only thanks to their inferences, but also because of the meaning of their non-logical vocabulary. `WE` has the advantage of ensuring monotonicity but, since atomic systems fix non-logical meaning, expanding atomic systems in the open case implies changing this meaning, so `WE` has also the shortcoming of relaxing too much the idea that validity depends on given non-logical features.

This sort of priority of the "logico-deductive" aspects of validity is also a shortcoming of $\models_2$. $\Gamma \models_2 A$ means that we have an argument structure from Γ to A whose non-introductory inferences are (provably) justified based on *one and the same* (recursive) set of functions over every systems, abstracting from how justification procedures interact with atomic rules. Our argument is logically valid, not (only) in the sense of being justifiable for every determination of non-logical meaning, but in the sense of being justified following the same pattern throughout variations of this meaning. This stability of validity is similar to the monotonic character of logical validity, granted by `WE`. In contrast, $\models_1$ looks at logical validity in a much more "model-theoretic" sense, i.e. as justifiability relative to determinations of non-logical meanings. This is a natural generalisation of local `NE`-validity, and accordingly one where "universal" validity is not correctness of justified inferences irrespective of non-logical meanings, but adaptability of these inferences to those meanings.

If one accepts this reconstruction, one may be also tempted to conclude that, if we adopt `NE` (resp. `WE`) at the local level, we should then adopt $\models_1$ (resp. $\models_2$) at the "global" level. However, as a second aim of my talk, I suggest that the aforementioned symmetries stem from a deeper duality in Prawitz's semantics, i.e. meaning of non-logical terminology *vs* justification of generalised eliminations. In particular, I maintain that Prawitz's notions of validity are based on four basic "ingredients", concerning the level (local or global), the focus and the abstraction elements (atomic bases and/or justification procedures), and the scope (one-at-a-time or class-like) where the focus element ranges, within the validity definitions.

Based on this "decomposition" of Prawitz's notions of validity, one can not only show that the "mixed" readings `NE` + $\models_2$ and `WE` + $\models_1$ enjoy a kind of symmetry too (inversion of the focus element), but additionally that the four readings highlighted so far constitute a diagram where certain interesting order relations hold, and which is complete, namely, the diagram provides a complete classification of proof-theoretic semantics compatible with Prawitzian tenets.

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Reduced Routley-Meyer models and labelled sequent calculi for relevant logics.

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Relevant logics are a well-known family of non-classical logics introduced to cope with so-called paradoxes of material implication. According to relevantists, $\to$ is intended to express a more fine-grained and philosophically motivated notion of conditional (see, amongst others, [9, 6, 2]). Part of the philosophical intuition of relevant logics, at least in the early development by Anderson and Belnap [1], was that the antecedent and consequent of a valid conditional must be relevant to each other, in the sense that, in expressions of the form $A \to B$, there must be a strong connection between antecedent and consequent.

The purpose of this talk is to perform a proof theoretic investigation of a wide number of relevant logics, by starting from the base system known under the label **B**. We will employ the well-established methodology of labelled sequent calculi, i.e., we will work with structures that internalize semantic informations within sequents while being capable, at the same time, of preserving several desirable proof-theoretic properties (see, for example, [5, 7]). At the semantic level, we will characterise relevant logic **B** and some of its extensions by using *reduced* Routley-Meyer models, namely, relational structures with a ternary relation between worlds along with a unique distinct element considered as the *real* (or *actual*) world (see, e.g., [9, 10, 3]). We will introduce a variety of labelled calculi that reflect, at the syntactic level, semantic informations taken exactly from reduced Routley-Meyer models. To be precise, rules for negation, $\sim$, and implication, $\to$, will have the following shape:

$$\frac{\Gamma \Rightarrow \Delta, a^* : A}{a : {\sim}A, \Gamma \Rightarrow \Delta}\ L{\sim} \qquad \frac{a^* : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, a : {\sim}A}\ R{\sim}$$

$$\frac{Rabc, a : A \to B, \Gamma \Rightarrow \Delta, b : A \qquad Rabc, c : B, a : A \to B, \Gamma \Rightarrow \Delta}{Rabc, a : A \to B, \Gamma \Rightarrow \Delta}\ L\to$$

$$\scriptstyle(b, c \text{ fresh})\ \displaystyle\frac{Rabc, b : A, \Gamma \Rightarrow \Delta, c : B}{\Gamma \Rightarrow \Delta, a : A \to B}\ R\to$$

The systems will also include rules acting on labels, i.e., on the objects of the form *Rabc*, as well as rules for disjunction $\vee$ and conjunction $\wedge$. Finally,

we will discuss some central results including soundness, completeness and CUT-admissibility.
In the second and final part of the talk, we will consider some advantages, as well as some weak points, of proof theoretic studies done via labelled sequents and provide some comparisons with other related works. Special attention will be given to the approaches of Viganò [11] and Negri, Kurokawa [4]. We shall conclude the talk by giving some indications for future research.

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The reverse mathematical strength of hyperations

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Hyperations have been introduced in [1] as a way to transfinitely iterate normal, i.e., strictly increasing continuous, functions on ordinals, refining the notion of Veblen functions. The goal of this talk is to discuss the existence of hyperations in the light of reverse mathematics. The formulation of this principle in second order arithmetic employs the notion of dilators introduced by Girard. These are particularly uniform transformations of linear orders, preserving well-foundedness. I will briefly outline how this framework allows to extend a sufficiently uniform categorical treatment of finite iterations to transfinite exponents. Such a construction has already appeared for the standard Veblen hierarchy in [2].
The main proof-theoretic discussion builds on a framework developed in [3], relating transfinitely iterated syntactic reflection to semantic ω-model reflection. The ordinal analysis of ATR_0 developed there is relativized to an arbitrary normal dilator T, yielding an equivalence between the principles

"*the hyperation of T preserves well-foundedness*"

and

$$\Pi^1_2\text{-}\omega\,\mathrm{RFN}\left(\Pi^1_1\text{-}\,\mathrm{BI} + \text{“}\,T \textit{ is a dilator}\text{”}\right)$$

over the weak base theory RCA_0. In particular, by Π^1_2-completeness of dilators, the uniform existence of hyperations is then equivalent to

$$\Pi^1_3\text{-}\omega\,\mathrm{RFN}\left(\Pi^1_1\text{-}\,\mathrm{BI}\right).$$

The master thesis on which this talk is based has been supervised by Andreas Weiermann and Fedor Pakhomov from the Logic group at Ghent University. It was also presented at the ASL Logic Colloquium 2022.

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A cyclic proof system for Guarded Kleene Algebra with Tests

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Kleene Algebra with Tests (KAT) is a system for reasoning about program equivalence. It is a finite quasi-equational theory with two sorts, namely *programs* and a subset thereof consisting of *tests*, such that the programs form a Kleene algebra under the operations $(+, \cdot, *, 0, 1)$ and the tests form a Boolean algebra under the operations $(+, \cdot, ^-, 0, 1)$.

In terms of programming constructs, the operations $+, \cdot, *$ respectively capture non-deterministic choice, sequential composition and arbitrary repetition. The inclusion of tests allows one to express if-then-else statements and while loops.

Despite the gain in expressive power, the complexity of deciding KAT-equalities remains the same as for Kleene Algebra, *i.e.* it is PSPACE-complete. In [2] a fragment of KAT is identified which is computationally much more efficient, yet still reasonably expressive. This fragment, called Guarded Kleene Algebra with Tests (GKAT), is obtained by replacing the operations $+$ and * by their guarded counterparts $+_{(b)}$ and $^{(b)}$. In terms of KAT the guarded operations can be encoded as follows:

$$e +_{(b)} f \mapsto b \cdot e + \overline{b} \cdot f \qquad\qquad e^{(b)} \mapsto (b \cdot e)^* \cdot \overline{b}$$

In this talk we propose a cyclic proof system for GKAT. This system, named SGKAT, is inspired by the cyclic system in [1] for ordinary Kleene Algebra. Its rules are given on the next page. In each rule σ denotes a list of literals (*i.e.* primitive tests or their negations) and capital Greek letters denote lists of GKAT-expressions. A derivation is said to be a *proof* if every infinite branch contains infinitely many application of (b)-l.

In this talk we shall present the soundness and completeness of SGKAT with respect to the language model from [2]. Furthermore, we shall compare SGKAT to the original system in [1]. Of particular interest is that the succedents of SGKAT-sequents are lists rather than multisets of lists. Time permitting, we shall discuss the following questions of our ongoing research:

(1) What is the least possible complexity of proof search?

(2) Can SGKAT be used to prove the completeness of some algebraic axiomatisation of GKAT with respect to the language model?

$$
\frac{}{\sigma, 0, \Gamma \Rightarrow \Delta}\ 0\text{-}l
\qquad
\frac{\sigma, \Gamma \Rightarrow \Delta}{\sigma, 1, \Gamma \Rightarrow \Delta}\ 1\text{-}l
\qquad
\frac{}{\sigma, t, \bar{t}, \Gamma \Rightarrow \Delta}\ \bot\text{-}l
$$

$$
\frac{\sigma, b, \Gamma \Rightarrow \Delta \qquad \sigma, c, \Gamma \Rightarrow \Delta}{\sigma, b \vee c, \Gamma \Rightarrow \Delta}\ \vee\text{-}l
\qquad
\frac{\sigma, e, g, \Gamma \Rightarrow \Delta}{\sigma, e \cdot g, \Gamma \Rightarrow \Delta}\ \cdot\text{-}l
$$

$$
\frac{\sigma, b, e, \Gamma \Rightarrow \Delta \qquad \sigma, \bar{b}, e, \Gamma \Rightarrow \Delta}{\sigma, e +_b f, \Gamma \Rightarrow \Delta}\ +_b\text{-}l
$$

$$
\frac{\sigma, b, e, e^{(b)}, \Gamma \Rightarrow \Delta \qquad \sigma, \bar{b}, \Gamma \Rightarrow \Delta}{\sigma, e^{(b)}, \Gamma \Rightarrow \Delta}\ (b)\text{-}l
$$

$$
\frac{\Gamma \Rightarrow \sigma, \Delta}{\Gamma \Rightarrow \sigma, 1, \Delta}\ 1\text{-}r
\qquad
\frac{\Gamma \Rightarrow \sigma, e, f, \Delta}{\Gamma \Rightarrow \sigma, e \cdot f, \Delta}\ \cdot\text{-}r
$$

$$
\frac{b, \Gamma \Rightarrow \sigma, b, \Delta \qquad \bar{b}, \Gamma \Rightarrow \sigma, c, \Delta}{\Gamma \Rightarrow \sigma, b \vee c, \Delta}\ \vee\text{-}r_1
\qquad
\frac{c, \Gamma \Rightarrow \sigma, c, \Delta \qquad \bar{c}, \Gamma \Rightarrow \sigma, b, \Delta}{\Gamma \Rightarrow \sigma, b \vee c, \Delta}\ \vee\text{-}r_2
$$

$$
\frac{b, \Gamma \Rightarrow \sigma, e, \Delta \qquad \bar{b}, \Gamma \Rightarrow \sigma, f, \Delta}{\Gamma \Rightarrow \sigma, e +_b f, \Delta}\ +_b\text{-}r
\qquad
\frac{b, \Gamma \Rightarrow \sigma, e, e^{(b)}, \Delta \qquad \bar{b}, \Gamma \Rightarrow \sigma, \Delta}{\Gamma \Rightarrow \sigma, e^{(b)}, \Delta}\ (b)\text{-}r
$$

$$
\frac{\Gamma \Rightarrow \Delta}{p, \Gamma \Rightarrow p, \Delta}\ \mathsf{k}
\qquad
\frac{}{\Rightarrow}\ \mathsf{id}
\qquad
\frac{b, \Gamma \Rightarrow \Delta}{b, \Gamma \Rightarrow b, \Delta}\ \mathsf{w}\text{-}r
\qquad
\frac{\Gamma \Rightarrow \Delta}{b, \Gamma \Rightarrow \Delta}\ \mathsf{w}\text{-}l
$$

$$
\frac{\Gamma \Rightarrow \sigma, b, b, \Delta}{\Gamma \Rightarrow \sigma, b, \Delta}\ \mathsf{c}\text{-}r
\qquad
\frac{\Gamma \Rightarrow \sigma, c, b, \Delta}{\Gamma \Rightarrow \sigma, b, c, \Delta}\ \mathsf{e}\text{-}r
\qquad
\frac{\sigma, b, b, \Gamma \Rightarrow \Delta}{\sigma, b, \Gamma \Rightarrow \Delta}\ \mathsf{c}\text{-}l
\qquad
\frac{\sigma, c, b, \Gamma \Rightarrow \Delta}{\sigma, b, c, \Gamma \Rightarrow \Delta}\ \mathsf{e}\text{-}l
$$

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Propositional Model Existence theorem's formalisation in Isabelle/HOL

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Logic's purpose is about knowledge's formalisation and its reasoning. This work is the continuation of *Elementos de lógica formalizados en Isabelle/HOL* [1] in which Syntax, Semantics and the propositional version of Hintikka's lemma are studied from the theoretical perspective of *First-order Logic and Automated Theorem Proving* [2] by Melvin Fitting. Following the same perspective, this project focuses on the demonstration of Propositional Model Existence theorem, concluding with the Propositional Compactness theorem as a consequence. Inspired by *Propositional Proof Systems* [3] by Julius Michaelis and Tobias Nipkow, these results will be formalised using Isabelle: a proof assistant including automatic reasoning tools to guide the user on formalising, verifying and automating results. In particular, Isabelle/HOL is the specialization of Isabelle for High-Order Logic. The formalisation in Isabelle/HOL of the results presented in this work follows two directions. In the first place, each lemma is proved in detail without any automation tool as the result of an inverse research on every step of the demonstration in order to reach a proof based on elementary rules and definitions. Conversely, an alternative proof using all the automatic reasoning tools that are provide by the proof assistant will be exposed. Isabelle's power of automatic reasoning is shown throughout this work as the contrast between these two opposite proving tactics.

The project's code is available on *GitHub* through the following link:
https://github.com/sofsanfer/TFM

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‘Provability Implies Provable Provability’ in FLINSPACE

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We create a theory of arithmetic for the class $\mathsf{FLINSPACE}$ (this class is the same as Grzegorczyk's class $\mathcal{E}^2$) that we call G_2. Our approaches are distinct from what is found in the literature, in particular the theory that we develop for $\mathsf{FLINSPACE}$ includes $\mathsf{I}\Delta_0$.

We explore connections between G_2 and $\mathsf{I}\Delta_0$, and their implications in the study of complexity classes. After that, we express the usual metamathematical notions in G_2: we define numerations of the axioms of a theory; we define the *standard proof predicate* $\mathsf{Prf}_\xi(x, y)$ that expresses “y is the code of a proof of the formula coded by x according to the numeration ξ”; and we define the *standard provability predicate* $\mathsf{Pr}_\xi(x) := \exists y.\mathsf{Prf}_\xi(x, y)$.

It is a known fact that the derivability condition ‘provability implies provable provability’ is very sensitive to the considered theory: more precisely, we have no guarantee that it holds for weak theories of arithmetic (it is an open problem for general numerations in $\mathsf{I}\Delta_0$).

We study the uniform derivability condition $\mathsf{Pr}_\xi(x) \to \mathsf{Pr}_\xi(\ulcorner \mathsf{Pr}_\xi(\dot{x}) \urcorner)$. We prove that if $\mathsf{Pr}^{\mathcal{S}}(x)$ is a provability predicate for a finite set of axioms $\mathcal{S}$ (including a finite number of logical axioms), then $\mathsf{G}_2 \vdash \mathsf{Pr}^{\mathcal{S}}(x) \to \mathsf{Pr}_\xi(\ulcorner \mathsf{Pr}_\xi(\dot{x}) \urcorner)$. Moreover, if G_2 can verify its axioms, in the sense that, for a suitable G_2-function $\mathsf{verifier}$, $\mathsf{G}_2 \vdash \xi(x) \to \mathsf{Prf}_\xi(\ulcorner \mathsf{Pr}_\xi(\dot{x}) \urcorner, \mathsf{verifier}(x))$, then $\mathsf{G}_2 \vdash \mathsf{Pr}_\xi(x) \to \mathsf{Pr}_\xi(\ulcorner \mathsf{Pr}_\xi(\dot{x}) \urcorner)$. A sufficient condition for the internal Σ_1-completeness of G_2 is also presented. Finally, we present conditions for a numeration ξ_0 of a finitely axiomatizable theory to satisfy $\mathsf{G}_2 \vdash \mathsf{Pr}_{\xi_0}(x) \to \mathsf{Pr}_\xi(\ulcorner \mathsf{Pr}_{\xi_0}(\dot{x}) \urcorner)$.

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On the uniform strength of some additive Ramseyan principles

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In this talk, we will give a characterization, in terms of Weihrauch degrees, of the uniform strength of some combinatorial principles related to Ramsey's theorem: the theorems we will be chiefly interested in assert the existence of almost-homogeneous sets for colourings of pairs of natural and rational numbers satisfying some properties determined by some additional algebraic structure on the set of colours.

In [1], it was shown that the principles above are equivalent, over RCA_0, to $\mathsf{I}\Sigma^0_2$. We will see that the analysis of the Weihrauch degrees of these theorems is somewhat more nuanced. The principles LPO' (the jump of the Limited Principle of Omniscience) and $\mathsf{TC}_\mathbb{N}$ (the total continuation of closed choice on $\mathbb{N}$) will be important charachters in this study.

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Proof-Theoretic Validity and Intuitionistic Logic

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This work demonstrates the effect of recent work on proof-theoretic validity on the arguments Dummett (1991) put forward in the Logical Basis of Metaphysics. There Dummett argued that a proof-theoretic meaning theory could support intuitionism and certain antirealist metaphysical positions. It also aims to give some of the history leading up to these results.

In the Logical Basis of Metaphysics, Dummett argues that we can find answers to deep metaphysical questions about the world, the mind, and mathematical reality by constructing a satisfactory theory of meaning. Over half of the book constructs an inferentialist and proof-theoretic theory of meaning. And shows how this theory leads to anti-realist metaphysics. Dummett does not endorse this semantics as the correct one. But his sympathies lie with this proof-theoretic approach.

In the 3 decades since the publication of the book, proof-theoretic semantics has made great technical strides. This allows us to assess some of the claims Dummett made considering this new knowledge. These results appear to undermine Dummett's argument. But we will offer a response to this worry.

Word meanings in proof-theoretic semantics are proof rules. Dummett's proof-theoretic semantics has two interconnected approaches. The first requires that the set of proof rules for a connective be in harmony. This is often spelt-out as the requirement that introducing a term and then eliminating it proves nothing new. The second approach is proof-theoretic validity. Prawitz (1973) developed this semantics in the early seventies. Proof-theoretic validity separates potential proofs into those that are valid and those that are invalid. It does this based on a property like, but not identical to, normalisation. When we talk of the logic of proof-theoretic validity we mean the logic that includes all and only the valid proofs.

Our focus here will be on proof-theoretic validity, the more technical of the two notions. Because it is the one that turns out to have the most surprises in store. Prawitz conjectured, and Dummett hoped, that proof-theoretic validity would be intuitionistic. For Dummett, this is important because it is intuitionistic logic being the correct logic that leads to antirealist metaphysics. Initial results showed that, for subsets of the connectives, Prawitz's conjecture was correct. But it soon became clear that proof-theoretic validity had super intuitionistic features.

To show that Prawitz's conjecture was false took further work. Because Prawitz's initial definition had not clearly defined the meanings of the atomic formulas. Piecha and Schroeder-Heister (2019) showed that for any clarification of Prawitz's definition proof-theoretic validity was not intuitionistic. Rather it was superintuitionistic. In addition: Goldfarb (2016) showed that Dummett's definition was superclassical. Stafford (2021) showed that a clarification of Prawitz's system was inquisitive logic. Oliveira (2021) showed that if you focus on elimination rules it was intuitionistic. Sandqvist (2009) even showed that by fudging the treatment of 'or' you could get classical logic.

The topic considered here is what do all these results mean for Dummett's work. Let's consider the two closest to the definition of proof-theoretic validity Dummett offered. These are Goldfarb's result for a modification of Dummett's definition and Stafford's result for a modification of Prawitz's definition. On the face of it, these results undermine the project. Goldfarb proved that Dummett's definition has superclassical validities. This removes Dummett's definition from serious consideration. Stafford's result proves that Prawitz's definition (spelt out) aligns with inquisitive logic. Inquisitive logic is a well-studied logic. It is a semantics for the joint treatment of questions and assertions. But it is not a semantics for assertions alone because it is not closed under substitution and its closure under substitution isn't decidable. Worse yet the rules of Inquisitive logic are not harmonious. This means that Dummett's two notions come apart. These results appear to demonstrate the failure of the project in the Logical Basis of Metaphysics.

The treatment of the atomic formulas causes these results. Every treatment takes a side on whether you can prove a disjunction without proving one of the disjuncts. It is this opinionatedness of the atomic formulas that lead to inquisitive logic.

Atomic formulas' meanings are always proof rules. But different approaches to proof-theoretic validity disagree on what the proof-theoretic analogue of a model is. Prawitz's treatment of atomic formulas has evolved since the seventies. His more recent view holds that the analogue to a model is a static language or a set of proof rules. But his earlier view was that the analogue to a model was a stage in a dynamic language. The language was dynamic in the sense that it was modelled growing as more rules were added. But why wasn't a view considered where the analogue of a model wasn't a stage in a dynamic language but the entire dynamic language?

It turns out that if one does this then the resulting logic is intuitionistic. If this treatment is justified, perhaps Dummett is saved from the technical results knocking at his metaphysical door. We will, however, end with a note of caution about Dummett's move from proof-theoretic semantics to intuitionistic logic via proof-theoretic validity.

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Universal Proof Theory: Constructive Rules and Feasible Admissibility

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Universal proof theory is a recent project to study the generic behavior of proof systems in the same way that universal algebra studies the generic behavior of algebras. It has been started by investigating the connections between the form of the rules that a proof system has and the logical properties the system enjoys. So far, the connection between a relaxed version of analytic rules and different flavours of interpolation (Craig, Lyndon, uniform, etc) has been investigated for many substructural, intermediate and modal logics.

In this talk, we will expand this research line by studying the connection between the form of the primitive rules of a proof system and the rules the system admits. First, we will present a general form for the constructively acceptable intuitionistic modal rules, namely the ones that respect the constructive character of the intuitionistic ground. We call a sequent system only consisting of these rules or some basic modal rules a *constructive calculus*. Then, we will show that any constructive calculus stronger than **CK** and satisfying a mild technical condition, feasibly admits all Visser's rules. As the disjunction property is a special case of these admissible rules, the constructive character of the constructive calculi will also be justified. The proof-theoretical method we use is also quite interesting in its own right, as it does not need any sort of cut elimination to establish the admissibility. The feasibility of the extraction method then is just a by-product of this avoidance.

This type of connection between the form of the rules and the admissible rules of the corresponding logic has two types of applications. On the positive side, it uniformly proves the feasible admissibility of all Visser's rules in the usual sequent systems of a broad range of intuitionistic modal logics, including CK, CKT, $\mathsf{CK4}$, $\mathsf{CS4}$, $\mathsf{CS5}$, their Fisher-Servi versions, the intuitionistic modal logics for bounded depth and bounded width and the propositional lax logic. On the negative side, though, it shows that if an intuitionistic modal logic stronger than CK and satisfying a mild technical condition does not admit the Visser's rules or specially does not have the disjunction property, then it does not have a constructive calculus. As the class of constructive calculi is a very general class of sequent systems to consider, this negative result presents an interesting proof theoretical result about generic proof systems and their *non*-existence.

Decidable Parsing Algorithm for Categorial Grammar with Type-raising

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Categorial Grammar (CG) [1] is a formalism for representing natural language syntax. We assign a category to each word and a rule to each phrase. The category consists the two-directional arrows. For example, the verb phrase is represented by the category $\mathsf{NP}\backslash\mathsf{S}$ as the verb phrase takes a subject from the left-hand side, *e.g.*, 'He **walks**'. For another example, the adverb is represented by the category $\mathsf{ADJ}/\mathsf{ADJ}$ as the adverb takes an adjective from the right-hand side, *e.g.*, '**very** fast'. As these arrows correspond to the implication in Logic, we can use the theorem-proving approach to parse natural language by CG.

Combinatory Categorial Grammar (CCG) [2] is an extension of CG with the combinatory rules to analyze linguistics phenomena. One of the combinatory rules is the T combinator called *type-raising rule* to correspond with swapping a head in X-bar theory, where the head takes another component. For instance, X is raised to $Y/(X\backslash Y)$ by the T combinator. The verb phrase takes noun phrase, *i.e.*, we regard the verb phrase as a head $\mathsf{NP}\backslash\mathsf{S}$. By the T combinator, the noun phrase is a head $\mathsf{S}/(\mathsf{NP}\backslash\mathsf{S})$.

In CG and its variants, we parse a sentence by proving the theorem $\Gamma \vdash \mathsf{S}$ where Γ is a sequence of categories, *e.g.*, "He walks" is given by $\mathsf{NP}, \mathsf{NP}\backslash\mathsf{S} \vdash \mathsf{S}$. Since the natural language sentence is not commutative, we could not exchange the categories in Γ. Moreover, the sequent calculus of CG with the type-raising rule is defined as follows, where Roman letters are categories, and Greek letters are sequences of categories.

$$\frac{}{X \vdash X}\,Id \qquad \frac{}{X/Y,\, Y \vdash X}\,A> \qquad \frac{}{X,\, X\backslash Y \vdash Y}\,A<$$

$$\frac{}{Y \vdash (X/Y)\backslash X}\,T> \qquad \frac{}{X \vdash Y/(X\backslash Y)}\,T< \qquad \frac{\Gamma \vdash X \qquad \Sigma, X, \Delta \vdash Y}{\Sigma, \Gamma, \Delta \vdash Y}\,\mathrm{Cut}$$

As the non-axiom rule is only the cut rule, the proof is not cut-free. The non-cut-free proof is a problem for the decidability of the sequent calculus. Thus, it is also a problem for the parsing algorithm. Especially, T combinator is the hard rule for decidability. Hence, there is a limitation of usage [3] of the rule in most CCG parsing algorithms. For example, a parser allows the type raising only for the noun phrase. In the present paper, we eliminate the limitation of the type-raising rule by the proof-theoretic analysis.

First, we show the decidability of the parsing algorithm in CG without the type-raising rule. We represent the parsing tree of CG shown in Figure 1 as a sequent calculus proof shown in Figure 2. In each branch, the length of the antecedent increases for each cut rule if the antecedent is non-empty. The number of candidates of parsing tree with $n+1$ words is $\frac{1}{n+1} \cdot {}_{2n}\mathcal{C}_n$, which is a Catalan Therefore, we could decide whether the given sentence is grammatical or not in finite steps.

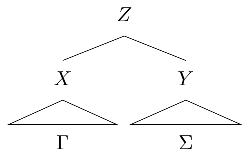

Figure 1: Tree of $\Gamma, \Sigma \vdash Z$

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\ \Gamma \vdash X\end{array} \quad \cfrac{}{X, Y \vdash Z}\,A< \text{ or } A>}{\Gamma, Y \vdash Z}\,\text{Cut} \qquad \begin{array}{c}\vdots\\ \Sigma \vdash Y\end{array}}{\Gamma, \Sigma \vdash Z}\,\text{Cut}$$

Figure 2: Proof of $\Gamma, \Sigma \vdash Z$

Next, we show the decidability of the parsing algorithm in CG with the type-raising rule. The parsing tree is formed by the cut rule and the type-raising rule, as shown in Figure 3. By the type-raising rule, we produce many candidates of trees. Thus, the above algorithm never halts if the sentence is ungrammatical. We here show the lemma: If a tree has unary rules in both branches, there is another parsing tree without the unary rules in one branch. The type-raising rule is swapping a head. Thus, to swap a head in both branches is 'swapping and swapping again.' At least, one swapping is redundant. By the lemma and the analysis of the complexity of the category (the number of / and \), we show the theorem: The number of type-rasing in Figure 3 is less than the maximum number of / and \ in the categories X and Y. Therefore, we prove each binary branch in finite steps. This is the decidability of the parsing algorithm in CG with the type-raising rule.

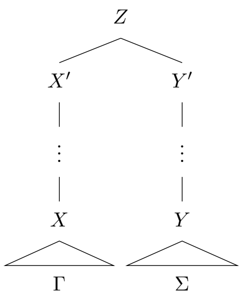

Figure 3: Tree of $\Gamma, \Sigma \vdash Z$ with T

Our contribution is two folds: In proof theory, we show the decidable algorithm to deduce s from a given sentence. In formal grammar, we remove the limitation of the type-raising rule in categorial grammar.

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Sequent Calculi for Two non-Fregean Theories

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The aim of the talk is to present Sequent Calculi for two non-Fregean theories: WB—a Boolean extension of the weakest non-Fregean logic SCI (*Sentential Calculus with Identity*), which was proposed by Roman Suszko [2], and WT—a topological extension of WB. Non-Fregean theories arose as a formalization of Wittgenstein's *Tractatus* [4], which was intertwined with the abolition of the (so called) Fregean Axiom [2]. Suszko disagreed with Fregean assumption that semantic correlates of sentences can be identified with their truth values and, instead, put into focus the concept of *situation* as the denotation of a sentence. This particular idea has been formalized by virtue of the introduction of the binary identity connective "$\equiv$", which is stronger than material equivalence "$\leftrightarrow$" and expresses the identity of situations denoted by two analyzed sentences.

WB is obtained from SCI through the addition of Boole algebra axioms, in which we expand properties of the identity connective. In WB we consider identity based on one introduced in SCI. However, WB consists of more tautological identities than SCI, where the only tautological identity was the trivial one of the form $\phi \equiv \phi$. In case of WB, $\phi \equiv \chi$ is a tautology if and only if $\phi \leftrightarrow \chi$ is a tautology of Classical Propositional Calculus. To formalize this notion we introduce proof system $\mathsf{G3}_{\mathsf{WB}}$ (based on a version of the original system $\ell\mathsf{G3}_{\mathsf{SCI}}$ found in [1]), in which each sequent is labelled with marker allowing (or disabling) the use of identity-dedicated rules. We will discuss correctness and invertibility of the proposed rule set and identify issues regarding the cut elimination procedure.

WT, its topological extension, differs from WB in the addition of supplementary identity-dedicated axioms. Its philosophical foundation lays in the following proposition from *Tractatus*:

> 5.141 If p follows from q and q from p then they are one and the same proposition.

which we can interpret as the fact that two logically equivalent sentences constitute different variants of the same sentence. Moreover, Suszko examined WT's correspondence to modal system S4 [3]. Identity can be thus interpreted through the means of modal necessity operator "$\Box$". We will briefly comment on this particular correspondence and then we will introduce $\mathsf{G3_{WT}}$, sequent calculus obtained from $\mathsf{G3_{WB}}$ as a result of two modifications: abandonment of the labels controlling application of the identity-dedicated rules and the weakening of the right-sided identity rule. Similarly as it was mentioned for $\mathsf{G3_{WB}}$, we will discuss its semantic status as well as the predictions regarding the cut elimination procedure.

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A Tree-Sequent Method for Intermediate Predicate Logic CD Expanded with Empirical Negation

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This presentation studies the addition of empirical negation to intermediate predicate logic $\mathbf{CD}$ and provides a *cut-free* tree-sequent calculus (a variant of nested sequent calculus) for this logic, denoted by $\mathbf{CD}^{\sim}$. [1] Although empirical negation was already added to intuitionistic [1, 2] and subintuitionistic [3] logic at the propositional level, it has not been in the first-order level. A sequent calculus for the logic obtained by adding empirical negation to intuitionistic propositional logic is proposed in [4], but this calculus is not cut-free. As is noted in [1, 2, 3], the satisfaction relation for empirical negation, denoted by "$\sim$", is defined as follows:

$$w \models_M \sim A \quad \text{iff} \quad g \not\models_M A,$$

where M is an intuitionistic Kripke model, w is a possible world in M, and g is the least element in the set of possible worlds in M. As is seen in this definition, empirical negation is formulated on a rooted Kripke model. Accordingly, the definition of semantic consequence is defined as the truth preservation on a root g.

The logic $\mathbf{CD}$ is obtained by adding to intuitionistic predicate logic the following axiom: $\forall x(A \vee B) \rightarrow (A \vee \forall x B)$, where x is not free in A. From a semantic viewpoint, a Kripke model for $\mathbf{CD}$ is obtained from an intuitionistic Kripke model by changing the condition of increasing domain to that of constant domain.

This presentation consists of three sections. The first section introduces Kripke semantics and Hilbert system for $\mathbf{CD}^{\sim}$ satisfying the soundness. A Kripke model for $\mathbf{CD}^{\sim}$ is obtained by adding to a Kripke model for $\mathbf{CD}$ the satisfaction relation for empirical negation, described above. The Hilbert system for $\mathbf{CD}^{\sim}$ is obtained by adding to $\mathbf{CD}$ the following axioms and rules related to "$\sim$":

$$A \vee \sim A,$$
$$\sim A \rightarrow (\sim\sim A \rightarrow B),$$
$$\text{From } A \vee B, \text{ we may infer } \sim A \rightarrow B,$$
$$\text{From } \sim A[z/x] \vee B, \text{ we may infer } \sim\sim \exists x A \rightarrow B.$$

Both of the axioms and the first rule were already provided in [2].

The second section provides tree-sequent calculus $\mathbf{T}\mathrm{CD}^{\sim}$ for $\mathbf{CD}^{\sim}$ and shows the strong completeness. A tree-sequent calculus handles a tree-sequent, which

[1] This work is based on discussion with Katsuhiko Sano, the supervisor of the author.

expresses the tree of sequents structured by labels. A *label* is a finite sequence of natural numbers $\langle n_1, \ldots, n_k \rangle$. If $\alpha = \langle n_1, \ldots, n_k \rangle$, then $\alpha \cdot n$ is the label $\langle n_1, \ldots, n_k, n \rangle$. A *tree* is the set of labels $\mathcal{T}$ such that $\langle\rangle \in \mathcal{T}$ and $\alpha \in \mathcal{T}$ for each $\alpha \cdot n \in \mathcal{T}$. A *labelled formula* is a pair $\alpha : A$ where α is a label and A is a formula. A *tree-sequent* is an expression $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$ where Γ and Δ are finite sets of labelled formulas, $\mathcal{T}$ is a tree, and all labels appearing in Γ and Δ are elements of $\mathcal{T}$. The calculus $\mathbf{T}\mathsf{CD}^{\sim}$ is obtained by adding to $\mathbf{T}\mathsf{CD}$, the calculus for $\mathbf{CD}$, provided in [5, 6], the following rules for empirical negation:

$$\frac{\langle\rangle : A, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha :\sim A}\ (\Rightarrow\sim) \quad \frac{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \langle\rangle : A}{\alpha :\sim A, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}\ (\sim\Rightarrow) .$$

The strong completeness of $\mathbf{T}\mathsf{CD}^{\sim}$ is shown without so much difficulty, since a tree-sequent reflects the structure of a Kripke model.

The third section defines the formulaic translation of a tree-sequent. Although the definition is based on the one in [6], a special treatment is needed in order to handle the label $\langle\rangle$. Based on the definition, the following theorem is obtained.

Theorem 1. For any set $\Gamma \cup \{A\}$ of formulas, the following are all equivalent:

- A is a semantic consequence of Γ,
- A is derivable from the assumptions Γ in the Hilbert system for $\mathbf{CD}^{\sim}$,
- A tree-sequent $\Gamma \overset{\mathcal{T}}{\Rightarrow} A$ is derivable in $\mathbf{T}\mathsf{CD}^{\sim}$.

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A reconstruction of Crispin Wright's strict finitistic logic: With the existence predicate

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'Strict finitism' is a constructive view obtained from intuitionism by replacing the notion of 'possibility in principle', on which intuitionism is based, with 'possibility in practice'. Among the literature, Wright's [3] stands out as it contains a sketch of formal systems of strict finitistic reasoning. We will present our reconstruction of his first-order predicate logic. The formulation is in the classical metatheory, as opposed to the strict finitistic metatheory that he used. A sound and complete pair of a Kripke semantics and a natural deduction system will be provided.

While Wright's original semantics is similar to that of **IQC**, we will incorporate the existence predicate (E) as in **IQCE** (cf. e.g. [2]). This is to treat quantification properly. A model will be a tree-like structure that represents all possible histories of actual verification by an agent; and Wright's negation stands for practical impossibility, i.e. $k \models \neg A$ iff $l \not\models A$ for all l. Hence $\neg P(a)$ can meaningfully hold at k even if object a is not in the domain of k. Thus quantification should range over the objects in the whole frame if the term is within the scope of $\neg$; otherwise, it should be restricted. E will denote the objects that 'exist' or are 'available' to the agent, in order to explicate this restriction.

Formally, we require the strictness of E: $k \models P(c)$ implies $k \models E(c')$ for any atomic P and subterms c' of any closed c. Also, after [1], we use 'constant domain' intuitionistic frames, and regard the extension of E at k as the domain of k. The forcing conditions of the quantifiers for existing objects are as in [1]: we set

- If x is not in the scope of $\neg$, $k \models \forall x A$ iff $k \models E(\overline{d}) \to A[\overline{d}/x]$ for all $d \in D$ (the constant domain); and $k \models \exists x A$ iff $k \models E(\overline{d}) \wedge A[\overline{d}/x]$ for some d.
- If x is in the scope of $\neg$, $k \models \forall x A$ iff $k \models \top \to A[\overline{d}/x]$ for all d; and $k \models \exists x A$ iff $k \models A[\overline{d}/x]$ for some d.

Wright's implication $A \to B$ means that if A holds in the future, so does B: $k \models A \to B$ iff for any $k' \geq k$ with $k' \models A$, there is a $k'' \geq k'$ such that $k'' \models B$. This is intuitionistic implication with 'time-gap'. He did not restrict the length of the gap. We assume it was because every structure in his strict finitistic metatheory is considered 'small enough'. We would, as part of classical idealisation, also accept a gap of any finite length.

As a result, $\neg A \vee \neg\neg A$, $((A \to B) \to A) \to A$ and $\sim\sim A \to A$ are found valid, where $\sim A$ is an abbreviation of $A \to \bot$. On the other hand, Modus Ponens $(A \to B, A/B)$ and $A \vee \neg A$ fail.

Our proof system comprises (i) all rules of **IQC**, and **IQCE**'s quantification rules and strictness rules with some modifications and (ii) $\neg$-introduction rules and (iii) a rule that takes care of the 'stability' of formulas. A is *stable* if $k \models \sim\sim A$ implies $k \models A$. We define a class ST of formulas with stability, and allow for inference $(\sim\sim S/S)$ for all $S \in$ ST. The completeness proof is in the Henkin-style.

Notably, Wright expected $(A \to B) \to ((A \to \neg B) \to \neg A)$ to be valid in his strict finitistic metatheory, but it is not in our logic. In fact in the classical formulastion, we find it equivalent to (i) $\neg\neg A \to A$ and to (ii) a semantic principle Wright rejected. We say a formula A is *prevalent* if for any k, there is a $k' \geq k$ with $k' \models A$. He rejected that every satisfiable formula is prevalent (the *formula prevalence property*), as it is unnatural: the verification of a formula may as well require so many resources that it could not be verified after verifying others.

Under the formula prevalence, two distinctions are lost: $k \models \neg A$ iff $k \models \sim A$, and A is satisfiable iff A is prevalent. If, further, $E(\overline{d})$ is prevalent for all $d \in D$ (the *object prevalence property*), then the distinction between the two modes of universal quantification is also lost: $k \models \forall x A$ iff $k \models \top \to A[\overline{d}/x]$. We will show some results that satisfiability and validity in the models with these prevalence properties behave almost classically.

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